The optimal time to adapt the processing rate in a Make-to-order production system

SMMSO 2017

Jannik Vogel, Raik Stolletz

Lecce, June 2017

When to react to increasing demand? Optimization of service systems with discretionary task completion

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Introduction

- Many tasks are completed according to subjective completion criteria
  * Discretionary tasks (Hopp et al., 2007)
  * Customer-intensive services (Anand et al., 2011)
- Health care, personal care, legal or financial consultancy, software engineering, call centers
- Tradeoff **Quality vs. Speed**
  * Fast service $\Longrightarrow$ Low quality ☹, Low waiting times ☺
  * Slow service $\Longrightarrow$ High quality ☺, High waiting times ☹
- Time-dependent setting
  * How are decisions influenced by demand changes in the future?

Agenda

1. Literature overview
2. Problem description & model formulation
3. Stationary solutions
4. Time-dependent results
   - Deterministic fluid approach
   - Stochastic SBC approach
5. Numerical results
6. Summary

Literature overview

1. Service rate decisions **without impact on quality**
   1.1 Based on current work-in-process (George and Harrison, 2001; Stidham and Weber, 1989)
   1.2 Based on demand over finite horizon (Parlar, 1984; Alam, 1979)

2. Service rate decisions **with impact on quality**

Paper	Queue	Objective		Demand	Dynamics
		Quality	Cong.		
Hopp et al. (2007)	$M/D/1$	Exponential	L^S	Ex.	Stat.
Wang et al. (2010)	$M/G/c$	Error prob.	W^Q	End.	Stat.
Anand et al. (2011)	$M/M/1$	Linear	W^S	End.	Stat.
Kostami and Rajagopalan (2014)	$M/M(t)/1$	-	L^S	End.	T-d.
Our model	$D(t)/D(t)/c$ $M(t)/M(t)/c$	Exponential	W^S	Ex.	T-d.

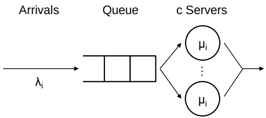
Cong. = Congestion measure; Ex. = Exogenous; End. = Endogenous; Stat. = Stationary; T-d. = Time-dependent

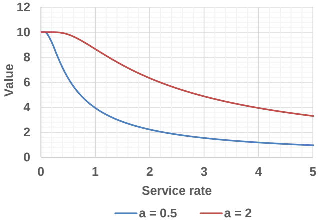
No publication considers time-dependent demand!

Agenda

Problem description

- $D(t)/D(t)/c$ and $M(t)/M(t)/c$ systems
- **Input:**
  - Arrival rates $\lambda_i \quad \forall i = 1, ..., n$
- **Decisions:**
  - Service rates $\mu_i \leq \overline{\mu} \quad \forall i = 1, ..., n$
- **Objective:**
  - Value: $v(\mu_i) = \bar{v}(1 - e^{-\frac{a}{\mu_i}})$ per served costumer (Hopp et al., 2007)
    * $\bar{v}$: Maximum value
    * a: Sensitivity to service rate
  - Waiting cost w per time unit spent in the system





$(\bar{v} = 10)$

Model formulation

1. **Time-dependent model:**

$$\max\ Z = \ell \sum_{i=1}^{n} \left[\bar{v}(1 - e^{-\frac{a}{\mu_i}}) \mathrm{E}[Th_i(\mu_1, ..., \mu_i)] - w\,\mathrm{E}[W_i^S(\mu_1, ..., \mu_i)] \right] \quad (1)$$

$$\text{s.t.} \qquad 0 < \mu_i \leq \overline{\mu} \quad \forall i = 1, ..., n \quad (2)$$

Remark:

- Numerical solution
- Analytical solution

Model formulation

1. **Time-dependent model:**

$$\max Z = \ell \sum_{i=1}^{n} \left[\bar{v}(1 - e^{-\frac{a}{\mu_i}}) \mathrm{E}[Th_i(\mu_1, ..., \mu_i)] - w\, \mathrm{E}[W_i^S(\mu_1, ..., \mu_i)] \right] \quad (1)$$

s.t.

$$0 < \mu_i \leq \overline{\mu} \quad \forall i = 1, ..., n \quad (2)$$

Remark:

- Numerical solution
- Analytical solution

2. **Stationary model:**

$$\max Z = \bar{v}(1 - e^{-\frac{a}{\mu}})\lambda - w\, \mathrm{E}[W^S(\mu)] \quad (3)$$

s.t.

$$0 < \mu \leq \overline{\mu} \quad (4)$$

Remark:

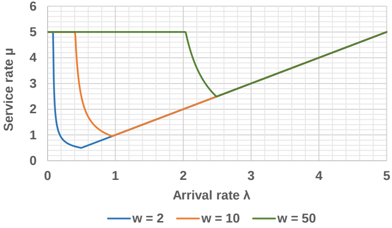
- A solution exists iff $\begin{cases} c\mu \geq \lambda & \text{deterministic model} \\ c\mu > \lambda & \text{stochastic model} \end{cases}$
- Analytical solution exists for $\begin{cases} c \geq 1 & \text{deterministic model} \\ c = 1 & \text{stochastic model} \end{cases}$

Agenda

Optimal solutions in the stationary system

***D/D/c*:**

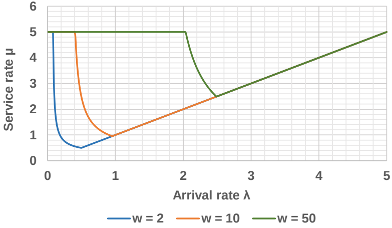
$$\mu^D = \begin{cases} \min\left(\max\left(\frac{a}{\ln\frac{a\bar{v}\lambda}{w}}, \frac{\lambda}{c}\right), \overline{\mu}\right) & \text{for } a\bar{v}\lambda > w \\ \overline{\mu} & \text{otherwise.} \end{cases}$$



Optimal solutions in the stationary system

D/D/c:

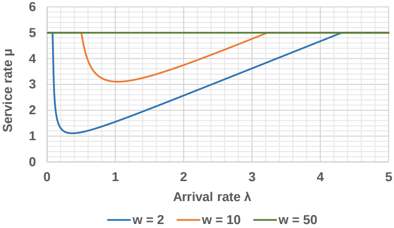
$$\mu^D = \begin{cases} \min\left(\max\left(\frac{a}{\ln \frac{a\bar{v}\lambda}{w}}, \frac{\lambda}{c}\right), \overline{\mu}\right) & \text{for } a\bar{v}\lambda > w \\ \overline{\mu} & \text{otherwise.} \end{cases}$$



M/M/1:

$$\mu^S = \begin{cases} \min\left(\mu^*, \overline{\mu}\right) & \text{for } a\bar{v}\lambda > w \\ \overline{\mu} & \text{otherwise} \end{cases}$$

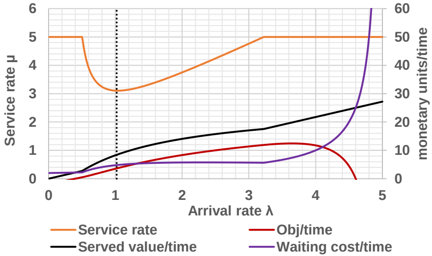
with μ^* determined by $\left(\frac{\mu^* - \lambda}{\mu^*}\right)^2 e^{-\frac{a}{\mu^*}} = \frac{w}{a\bar{v}\lambda}$.

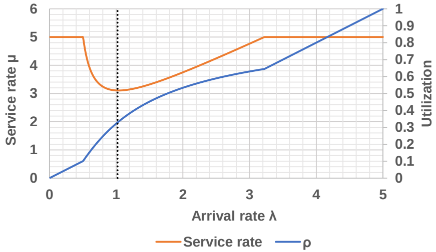


(Parameters in the figures: $c = 1, \bar{v} = 30, a = 1, \overline{\mu} = 5$)

Optimal solutions in the stationary system (II)

Setting: $M/M/1$, $c = 1, \bar{v} = 30, a = 1, w = 10, \overline{\mu} = 5$





Findings:

- $\mu^S \geq \mu^D$
- High sensitivity in λ
- μ^S and μ^D no monotone function in λ

Agenda

Optimal solutions in the deterministic model

Fluid assumptions (Newell, 1971)

1. **Numerical solution**
   Determined by solving the MINLP

2. **Analytical solution** for increasing demand λ_i

$$\mu_i^{TD} = \begin{cases} \min\left(\max\left(\frac{a}{\ln \frac{a\bar{v}\lambda_i}{w}}, \frac{\lambda_i}{c}\right), \overline{\mu}\right) & \text{for } a\bar{v}\lambda_i > w \\ \overline{\mu} & \text{otherwise.} \end{cases} \quad (5)$$

Optimal solutions in the deterministic model

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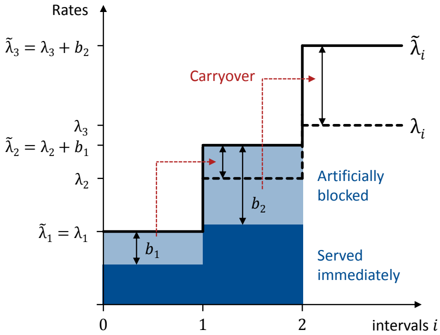
$$\mu_i^{TD} = \begin{cases} \min\left(\max\left(\frac{a}{\ln \frac{a\bar{v}\lambda_i}{w}}, \frac{\lambda_i}{c}\right), \overline{\mu}\right) & \text{for } a\bar{v}\lambda_i > w \\ \overline{\mu} & \text{otherwise.} \end{cases} \quad (5)$$

Finding: Optimal solution only depends on the current period!

Stochastic SBC approach (Stolletz, 2008)

For every period i with given length ℓ:

1. **Stationary loss system** $(M/M/c/c)$
   * Input: Artificial arrival rate $\tilde{\lambda}_i$
   * Output: Utilization $\mathrm{E}[U_i]$ and backlog rate b_i
2. **Stationary waiting system** $(M/M/c)$
   * Input: Modified arrival rate λ_i^{MAR} which results in $\mathrm{E}[U_i]$
   * Output: System performance



Good approximations for $\ell = \mu^{-1}$ $\Longrightarrow$ Calibration of the period length

NLP for the SBC approach

$$\max Z = \ell \sum_{i=1}^{n} \left[\bar{v}(1 - e^{-\frac{a}{\mu_i}}) \mathrm{E}[Th_i(\mu_1, ..., \mu_i)] - w\, \mathrm{E}[W_i^S(\mu_1, ..., \mu_i)] \right] \tag{6}$$

$$0 < \mu_i \leq \overline{\mu} \qquad \forall i = 1, ..., n \tag{7}$$

1. Step

$$b_0 = 0 \tag{8}$$

$$b_i = \tilde{\lambda}_i P_i^B \qquad \forall i = 1, ..., n \tag{9}$$

$$\tilde{\lambda}_i = \lambda_i + b_{i-1} \qquad \forall i = 1, ..., n \tag{10}$$

$$P_i^B = \frac{(\tilde{\lambda}_i/\mu_i)^c}{c! \sum_{k=0}^{c} \frac{(\tilde{\lambda}_i/\mu_i)^k}{k!}} \qquad \forall i = 1, ..., n \tag{11}$$

$$\mathrm{E}[U_i] = \frac{\tilde{\lambda}_i(1 - P_i^B)}{c\mu_i} = \frac{\lambda_i + b_{i-1} - b_i}{c\mu_i} \qquad \forall i = 1, ..., n \tag{12}$$

2. Step

$$\lambda_i^{MAR} = \mathrm{E}[U_i] c\mu_i = \lambda_i + b_{i-1} - b_i \qquad \forall i = 1, ..., n \tag{13}$$

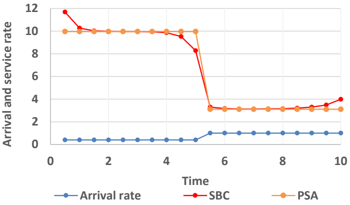
$$P_i^0 = \left(\sum_{n=0}^{c-1} \frac{(\lambda_i^{MAR}/\mu_i)^n}{n!} + \frac{(\lambda_i^{MAR}/\mu_i)^c}{c! \cdot (1 - \frac{\lambda_i}{c\mu_i})} \right)^{-1} \qquad \forall i = 1, ..., n \tag{14}$$

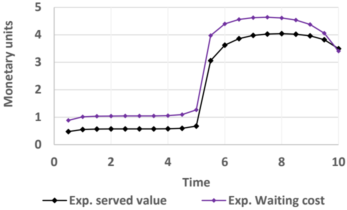
$$\mathrm{E}[W_i^S] = \frac{(\lambda_i^{MAR}/\mu_i)^c}{(c-1)!\mu_i(c - \lambda_i^{MAR}/\mu_i)^2} P_i^0 + \frac{1}{\mu_i} \qquad \forall i = 1, ..., n \tag{15}$$

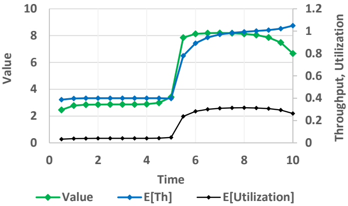
Agenda

Anticipation of demand changes: λ_i low

Setting: $c = 1$, $n = 20$, $\ell = 0.5$, $w = 10$, $\bar{v} = 30$, $a = 1$,
$\lambda(t) = 0.4$ for $t \leq 5, 1$ otherwise.







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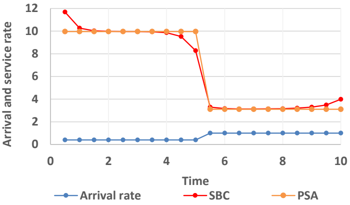
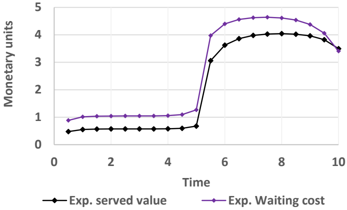
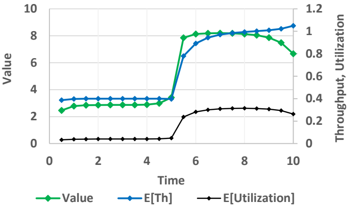


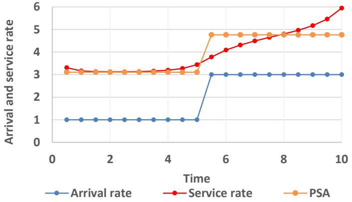


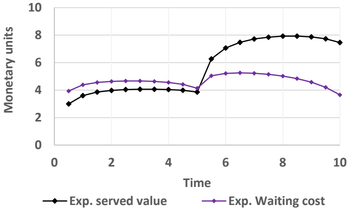


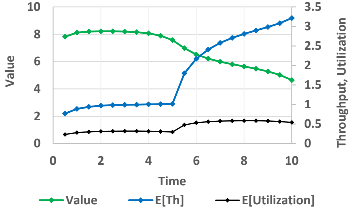
Finding: Service rate changes before the demand changes!

Anticipation of demand changes: λ_i high

Setting: $c = 1$, $n = 20$, $\ell = 0.5$, $w = 10$, $\bar{v} = 30$, $a = 1$,
$\lambda(t) = 1$ for $t \leq 5, 3$ otherwise.

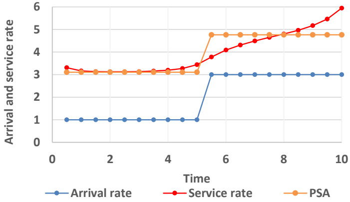


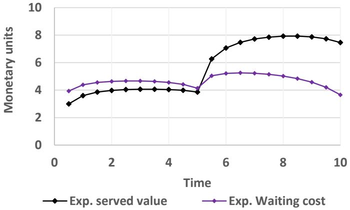


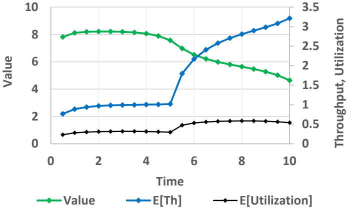


Anticipation of demand changes: λ_i high

Setting: $c = 1,\ n = 20,\ \ell = 0.5,\ w = 10,\ \bar{v} = 30, a = 1,$
$\lambda(t) = 1$ for $t \leq 5, 3$ otherwise.







Finding: Demand change leads to several service rate changes.

Summary

Conclusion:

- **Model:**
  - Quality-speed tradeoff
  - New model: Service rate optimization with time-dependent demand
- **Method:** Iterative procedure for the SBC-approach
- **Managerial insights**
  * Optimal service rate not monotone in λ
  * Deterministic model: Increasing demand does not influence service rates beforehand
  * Stochastic model: Later demand influences decisions

Summary

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- **Model:**
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  * Optimal service rate not monotone in λ
  * Deterministic model: Increasing demand does not influence service rates beforehand
  * Stochastic model: Later demand influences decisions

Future research:

- Include state-dependent information

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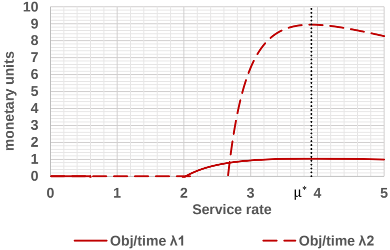
Objective value and performance measures depending on μ

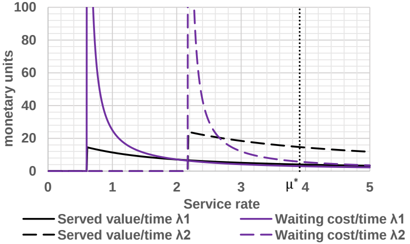
Setting: $c = 1, \bar{v} = 30, a = 1, w = 10, \overline{\mu} = 5, \lambda_1 = 0.6, \lambda_2 = 2.17, \mu^* = 3.91$

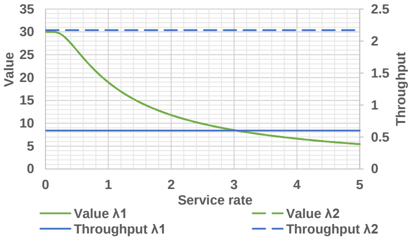
Condition for optimal μ:

$$ServedValue(\mu)' - WaitingCost(\mu)' = 0 \qquad (16)$$

$$\frac{a}{\mu^{*2}} \bar{v} \lambda e^{-a/\mu^*} = \frac{w}{(\lambda - \mu^*)^2} \qquad (17)$$







Optimal solutions for a $D/D/c$-system: Proof

$$\max Z = \bar{v}(1 - e^{-\frac{a}{\mu}})\lambda - \frac{w}{\mu} \tag{18}$$

Setting the first order partial derivative $\frac{\partial Z}{\partial \mu} = \frac{w}{\mu^2}(1 - \frac{a\bar{v}\lambda}{w}e^{-\frac{a}{\mu}})$ to zero gives $1 - \frac{a\bar{v}\lambda}{w}e^{-\frac{a}{\mu}} = 0$. Let us first assume that $a\bar{v}\lambda > w$. The zero of the function is found at $\mu' = \frac{a}{\ln \frac{a\bar{v}\lambda}{w}}$. Notice that for $\mu < \mu'$, $\frac{\partial Z}{\partial \mu}(\mu) > 0$ and for $\mu > \mu'$, $\frac{\partial Z}{\partial \mu}(\mu) < 0$. Thus, there is a maximum at μ'. Furthermore, if $\mu' < \frac{\lambda}{c}$, the objective function (18) is maximized at $\frac{\lambda}{c}$. If $\mu' > \overline{\mu}$, (18) is maximized at $\overline{\mu}$.

Let us now consider the special case $a\bar{v}\lambda \leq w$. For those parameters the zero of the first order partial derivative does not lie in A. Notice that $\frac{\partial Z}{\partial \mu}(\mu) > 0$ for $\mu \in A$. Thus, the maximum is attained for $\mu = \overline{\mu}$.

Optimal solutions for an $M/M/1$-system: Proof

$$\max Z = \bar{v}(1 - e^{-\frac{a}{\mu}})\lambda - \frac{w}{\mu - \lambda} \tag{19}$$

Setting the first order partial derivative equal zero gives $\left(\frac{\mu-\lambda}{\mu}\right)^2 e^{-\frac{a}{\mu}} = \frac{w}{a\bar{v}\lambda}$. Notice that the left hand side of the equation, $f : (\lambda, \infty) \to (0,1), \mu \mapsto \left(\frac{\mu-\lambda}{\mu}\right)^2 e^{-\frac{a}{\mu}}$ is continuous and strictly monotonically increasing. Therefore, for all $a\bar{v}\lambda > w$, a solution can be found. However, this solution does not need to lie in A and then the optimal solution is $\overline{\mu}$, because the first order partial derivative in μ is positive on the set $(\lambda, \overline{\mu})$.

Let us now consider the special case $a\bar{v}\lambda \leq w$. For those parameters the zero of the first order partial derivative does not lie in A. Notice that $\frac{\partial Z}{\partial \mu}(\mu) > 0$ for $\mu > \lambda$. Thus, the maximum is attained for $\mu = \overline{\mu}$.

Fluid approximation (Newell, 1971)

- **Key idea:** Replace discrete stochastic arrivals by a deterministic continuum
- Queue length at the end of period i:

$$\mathsf{E}[L_i^{Q,end}] = \max\{\mathsf{E}[L_{i-1}^{Q,end}] + \ell(\lambda_i - c\mu_i), 0\} \tag{20}$$

- Percentage of period i that has a positive queue length:

$$\pi_i = \begin{cases} 0 & \mathsf{E}[L_{i-1}^{Q,end}] = 0, \mathsf{E}[L_i^{Q,end}] = 0 \\ \frac{\mathsf{E}[L_{i-1}^{Q,end}]}{\ell(c\mu_i - \lambda_i)} & \mathsf{E}[L_{i-1}^{Q,end}] = 1, \mathsf{E}[L_i^{Q,end}] = 0 \\ 1 & \mathsf{E}[L_i^{Q,end}] > 0 \end{cases} \quad \forall i = 1, ..., n \tag{21}$$

- Expected average queue length in period i: $\mathsf{E}[L_i^Q] = \pi_i \frac{\mathsf{E}[L_{i-1}^{Q,end}] + \mathsf{E}[L_i^{Q,end}]}{2}$
- Expected average cycle time:

$$\mathsf{E}[W_i^S] = \begin{cases} \frac{1}{\mu_i} & \mathsf{E}[L_{i-1}^{Q,end}] = 0, \mathsf{E}[L_i^{Q.end}] = 0 \\ \mathsf{E}[L_i^Q]/\lambda_i + \frac{1}{\mu_i} & \mathsf{E}[L_{i-1}^{Q,end}] = 1, \mathsf{E}[L_i^{Q.end}] = 0 \\ \mathsf{E}[L_i^Q]/\mu_i + \frac{1}{\mu_i} & \mathsf{E}[L_i^{Q,end}] = 1 \end{cases} \tag{22}$$

MINLP for the fluid approach

$$\begin{aligned}
\mathrm{E}[L_0^{Q,end}] &= 0 && && (23)\\
\mathrm{E}[L_i^{Q,end}] &= \mathrm{E}[L_{i-1}^{Q,end}] + \ell(\lambda_i - \mathrm{E}[Th_i]) && \forall i && (24)\\
\mathrm{E}[Th_i] &\le c\mu_i && \forall i && (25)\\
c\mu_i - \mathrm{E}[Th_i] &\le M(1-\beta_i) && \forall i && (26)\\
\mathrm{E}[L_i^{Q,end}] &\le M\beta_i && \forall i && (27)\\
\beta_i &\le M\mathrm{E}[L_i^{Q,end}] && \forall i && (28)\\
\mathrm{E}[L_i^Q] &= \pi_i \frac{\mathrm{E}[L_{i-1}^{Q,end}] + \mathrm{E}[L_i^{Q,end}]}{2} && \forall i && (29)\\
\ell c\mu_i\pi_i &\ge (\mathrm{E}[L_{i-1}^{Q,end}] + \ell\lambda_i\pi_i)(1-\beta_i)\beta_{i-1} && \forall i && (30)\\
\pi_i &\ge \beta_i && \forall i && (31)\\
\pi_i &\le \beta_{i-1} + \beta_i && \forall i && (32)\\
\mathrm{E}[W_i^Q] &= \beta_{i-1}(1-\beta_i)\frac{\mathrm{E}[L_i^Q]}{\lambda_i} + \beta_i\frac{\mathrm{E}[L_i^Q]}{\mu_i} && \forall i && (33)\\
\mathrm{E}[W_i^S] &= \mathrm{E}[W_i^Q] + \frac{1}{\mu_i} && \forall i && (34)\\
\beta_i &\in \{0,1\} && \forall i && (35)\\
0 \le \pi_i &\le 1 && \forall i && (36)\\
\mathrm{E}[L_i^Q], \mathrm{E}[L_i^{Q,end}], \mathrm{E}[W_i^S], \mathrm{E}[Th_i] &\ge 0 && \forall i && (37)
\end{aligned}$$

Optimal solution in the deterministic system: Proof idea

Let j be the smallest period with $c\overline{\mu} < \lambda_j$, $n+1$ if non-existing.

(i) Consider periods $i = 1, ..., j-1$ sequentially.
- Assume $\mu_i < \frac{\lambda_i}{c} \implies$ Costumers left in the queue at the end of i
  * Leaving costumers in the queue (even tough they could have been served) cannot be optimal
  * Reducing additional queue later on is not optimal, because of the concave value-rate function
- Finding an optimal $\mu_i \geq \frac{\lambda_i}{c}$ leads to the same solution as in the stationary model.

(ii) Consider periods $i = j, ..., n$ sequentially. Objective function is increasing in the service rate μ_i. $\implies \overline{\mu}$ is optimal.

Iterative procedure for the SBC-approach

- Performance approximation quality depends on period length (Stolletz, 2008)
  - * Period length ≈ Processing time
- Optimization of processing rates $\implies$ Good period length not known a priori
- **Key idea:** Evaluation periods with period length $\ell_{eval} = \frac{\ell}{N_{eval}}$
  - * divide decision period into multiple periods ($N_{eval} > 1$) or
  - * unify multiple decision periods to a single larger period ($N_{eval} < 1$)
- Number of decisions remains the same!
- Choose ℓ_{eval} such that

$$\frac{\ell}{N_{eval}} = \ell_{eval} \approx \frac{1}{\bar{\mu}} = \left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)^{-1} \tag{38}$$

- **Iterative procedure**

  $N_{eval} \leftarrow 1$

  do

  $\mu_i \leftarrow$ solve problem using N_{eval}

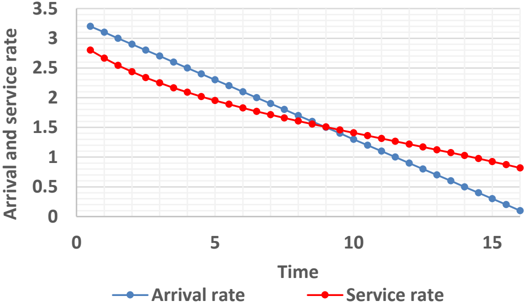
  $N_{eval} \leftarrow \ell\bar{\mu} = \frac{\ell}{n}\sum_{i=1}^{n}\mu_i$

  while N_{eval} different that in previous loop

Deterministic system with decreasing demand

Setting: $c = 1, n = 32, \ell = 0.5, w = 1, \bar{v} = 30, a = 1$

Optimal solution:

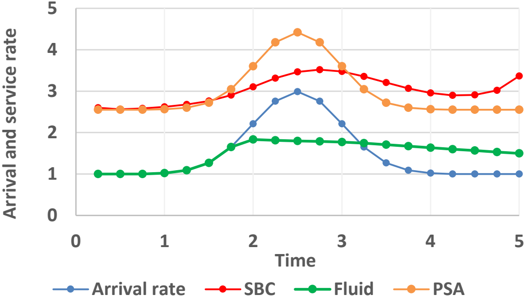


Comment: Anticipation of demand

The future does matter!

Impact of time-dependent decision

Setting: $\ell = 0.25$, $n = 20$, λ_i shows single peak, $w = 10$, $\bar{v} = 40$, $a = 1$

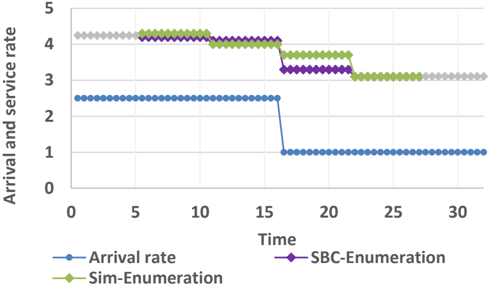


Objective value Z	Integrated model	Simulation	Difference
PSA	48.09	43.69	10.08%
Fluid	139.83	45.83	205.07%
SBC	48.35	46.26	4.52%

Optimal solution via simulation-enumeration

Setting: $\ell = 0.5$, $n = 64$, $\lambda(t)$ decreases at $t = 16$, $w = 10$, $\bar{v} = 30$, $a = 1$

(a) 4 decisions ($t = 5.5, 11, 16.5, 22$) (b) service rates: 2.8, ..., 4.6 (step size 0.1)



	Z	E[Th] total	E[W^S] total
Sim-Enumeration	216.05	55.64	16.35
SBC-Enumeration	217.85	55.76	16.85
SBC-Enumeration simulated	216.01	55.64	16.81
Difference (rel.)	0.02%	0.01%	2.75%