# On the Inter-departure, Inter-start, and Cycle Time Distribution of Closed Queueing Networks Subject to Blocking

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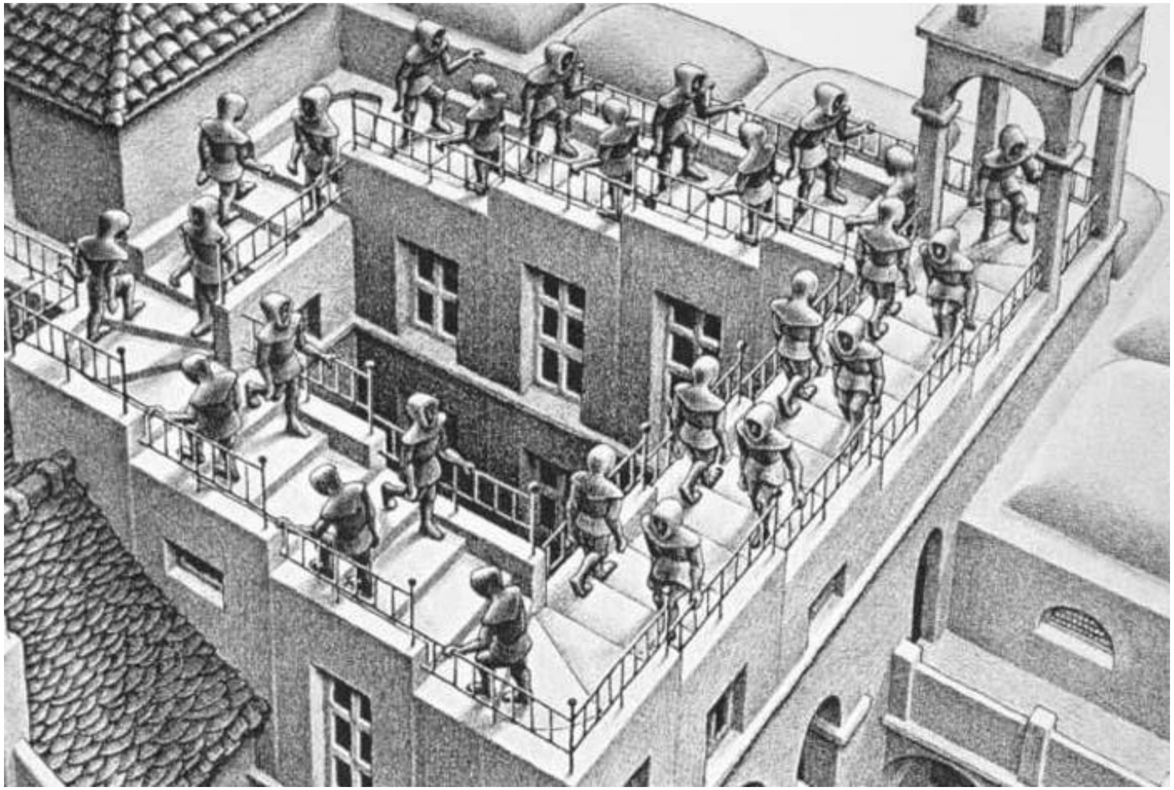
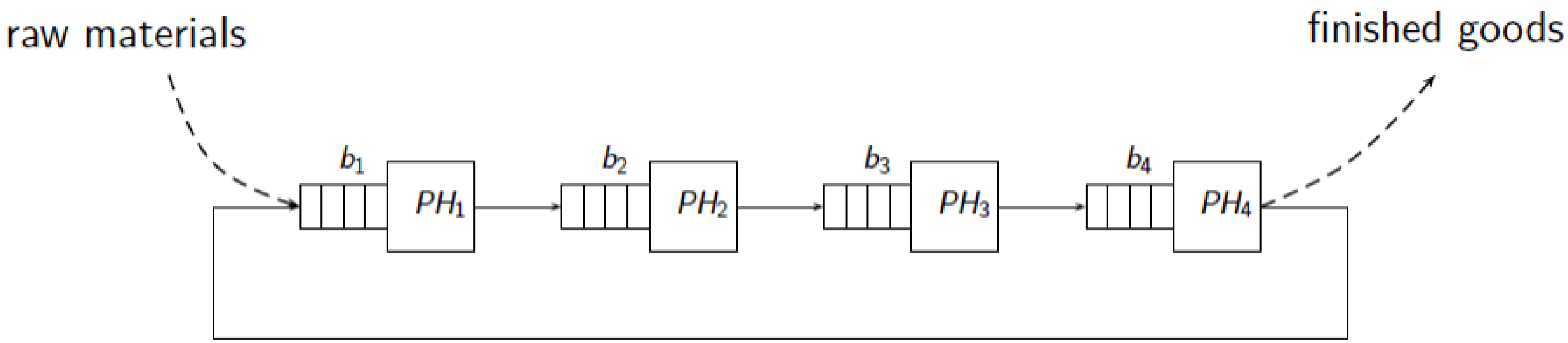


M. C. Escher, 1898 - 1972

# Plan

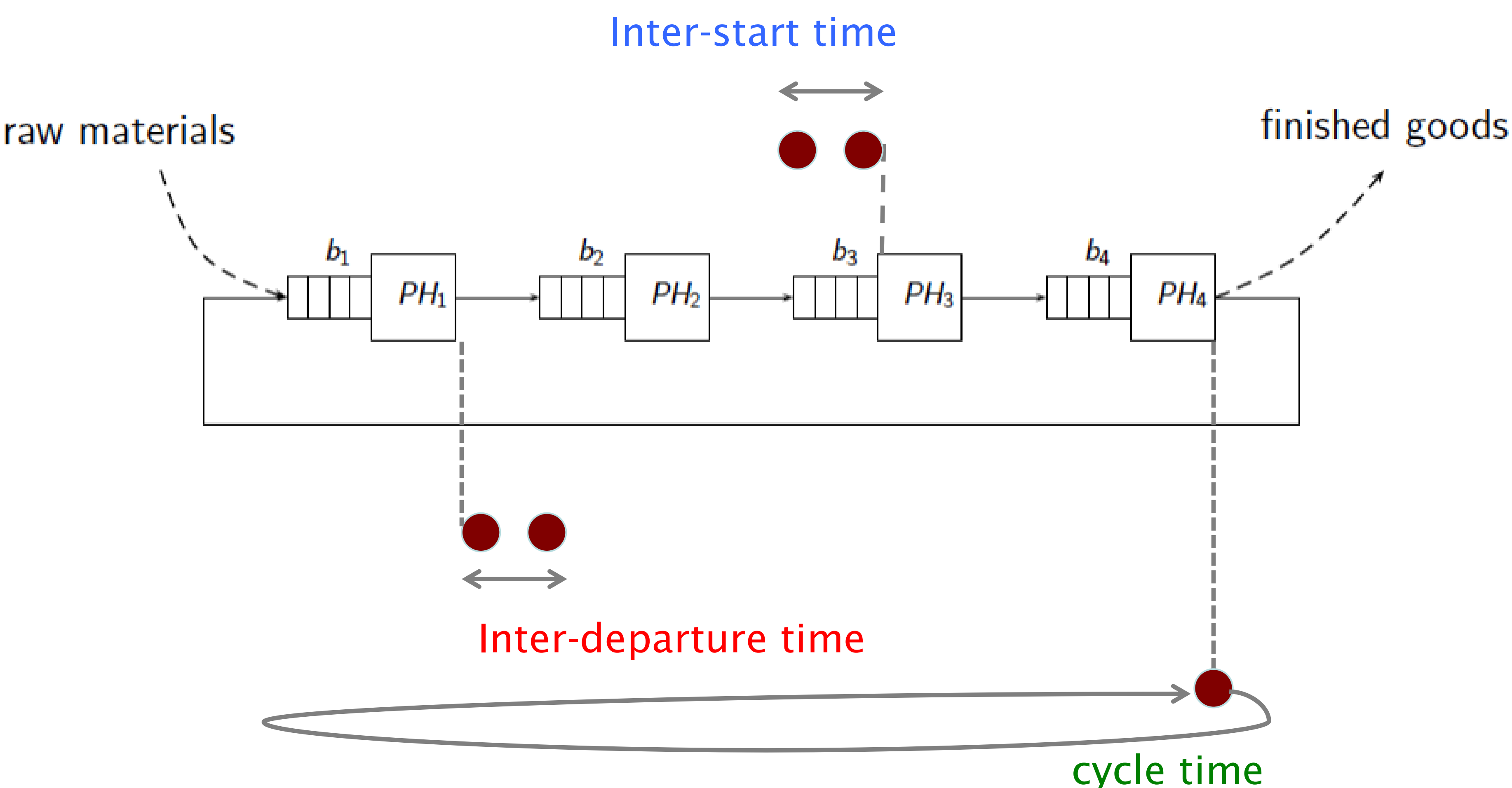
- ➡ Performance Evaluation of Closed Queueing Networks (CQN)
- Describing the Inter-event Times on an Extended State Space Originating from the CTMC Model of CQNs
- Determining the Inter-event Time Distribution as a First Passage Time Distribution on the Extended State Space
- Numerical Results

# Closed queueing networks are used for modeling production and telecommunication systems.




- closed queueing system
- linear flow of material
- finite buffer capacity
- phase-type distributed processing times
- no failures
- blocking after service
- first-come first-served queueing discipline
- one server per station

## The objective is to determine the **exact** distributions of the Inter-departure, Inter-start, and Cycle Time of Closed Queueing Networks Subject to Blocking




Exact results can be used to test the accuracy of approximation methods

# A method to determine the exact distribution of interdeparture, interstart, and cycle time distributions is not available in the literature

Variance of the inter-departure time or of the output:
Buzacott, Liu, and Shanthikumar (1995), Duenyas and Hopp (1990)
Duenyas, Hopp, and Spearman (1993), Gershwin (1993)
Gelenbe (1975), Hendricks (1992)
Hendricks and McClain (1993), Kim and Alden (1997)
Li and Meerkov (2000) , Miltenburg (1987)
Manitz (2005), Manitz and Tempelmeier (2012)
Tan (1999a), Tan (1999b), Tan (2000)

Inter-start time distribution:
Lagershausen (2012)

Cycle time distribution:
Chow (1980), Schassberger and Daduna (1983), Boxma and Donk (1982)
Boxma, Kelly, and Konheim (1984), Daduna (1982)
Kelly and Pollett (1983), Boxma (1983)

# Plan

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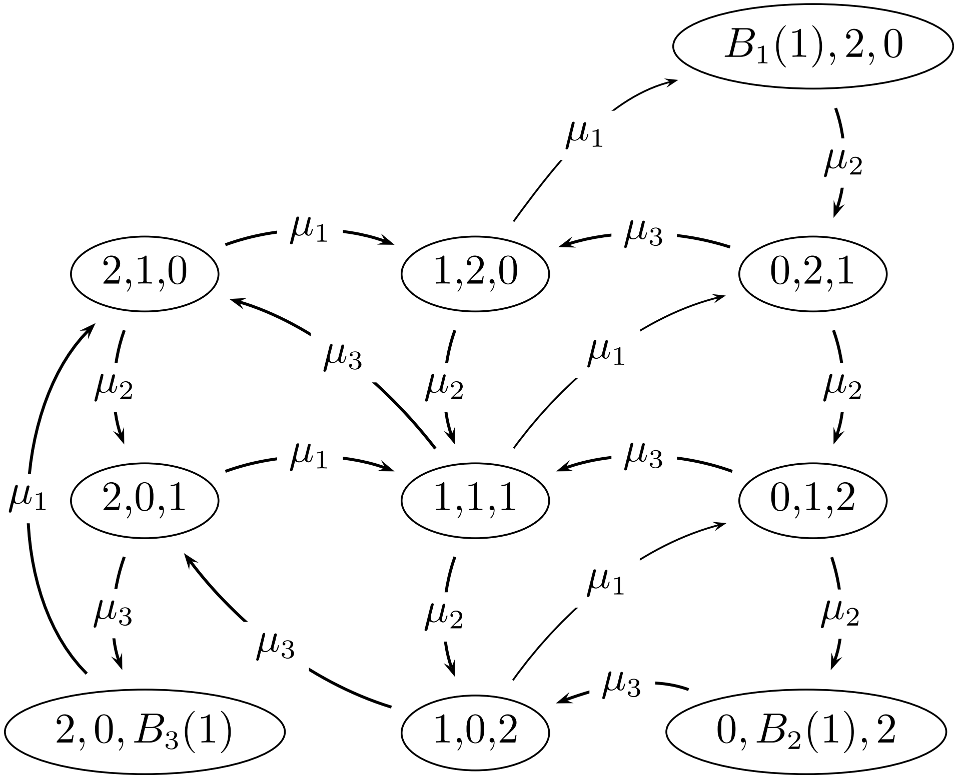
Numerical Results

# Our method is based on generating and analyzing the state transition matrix of the inter-event times from the original Markov chain model

3-station,
3-customer exponential CQN
with 1 buffer unit at each station:



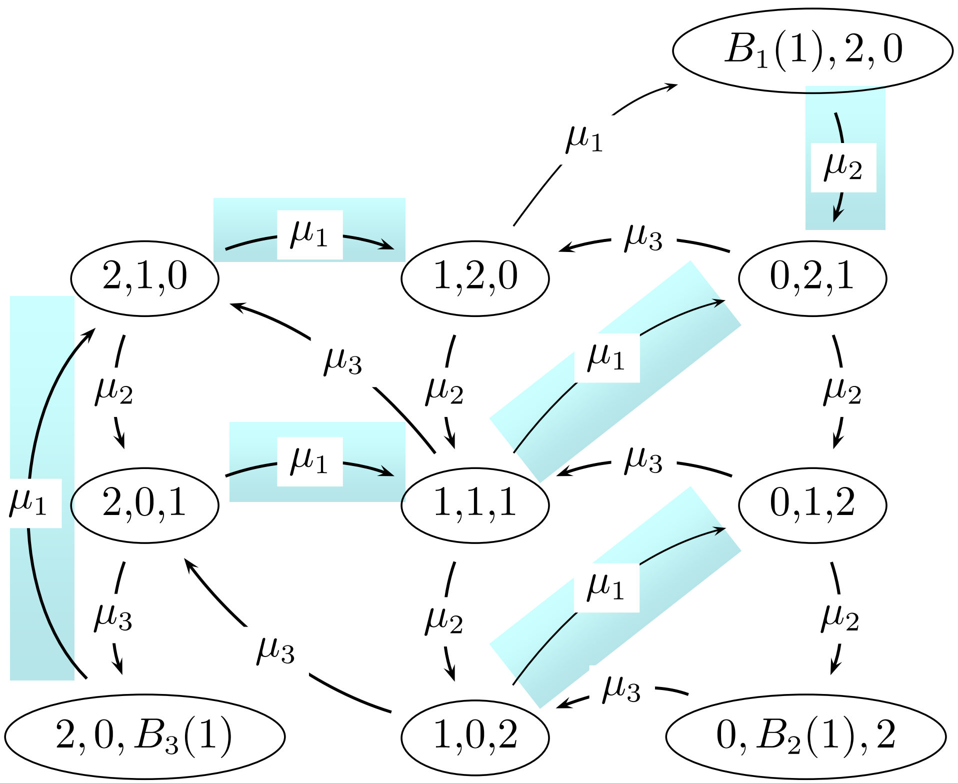

State description: (n_1, n_2, n_3)

n_i: number of
workpieces at station 1

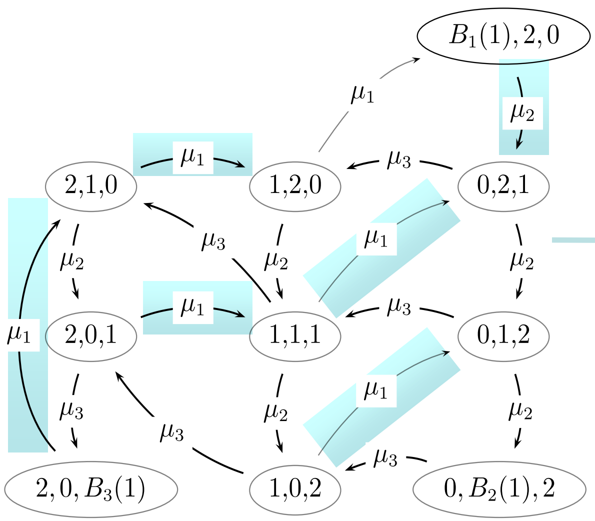
*(B_i(j): station I is blocked
while containing j workpieces)*

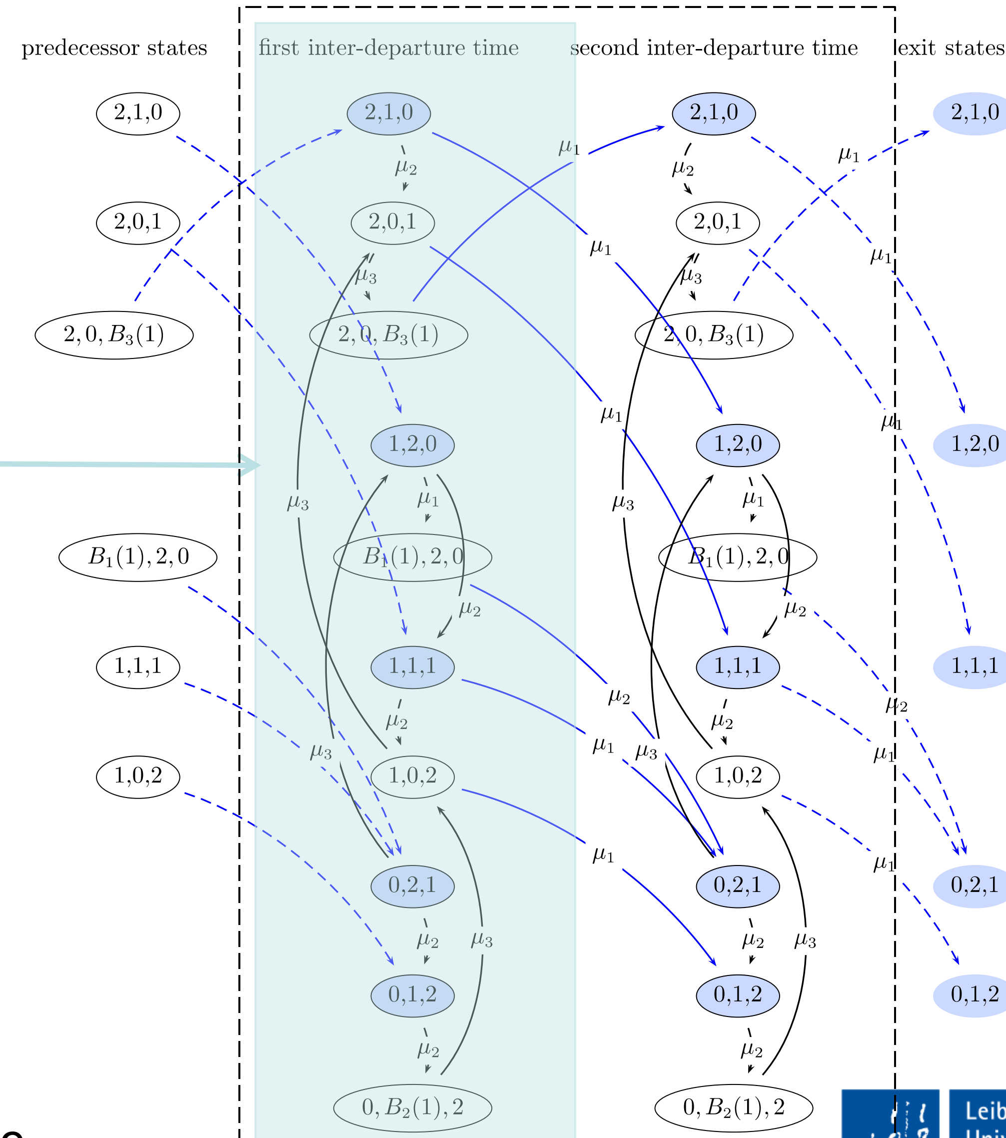
- Transition rate matrix of CTMC
- Steady-state probabilities

# Transitions with the event “Inter-departure at station 1”

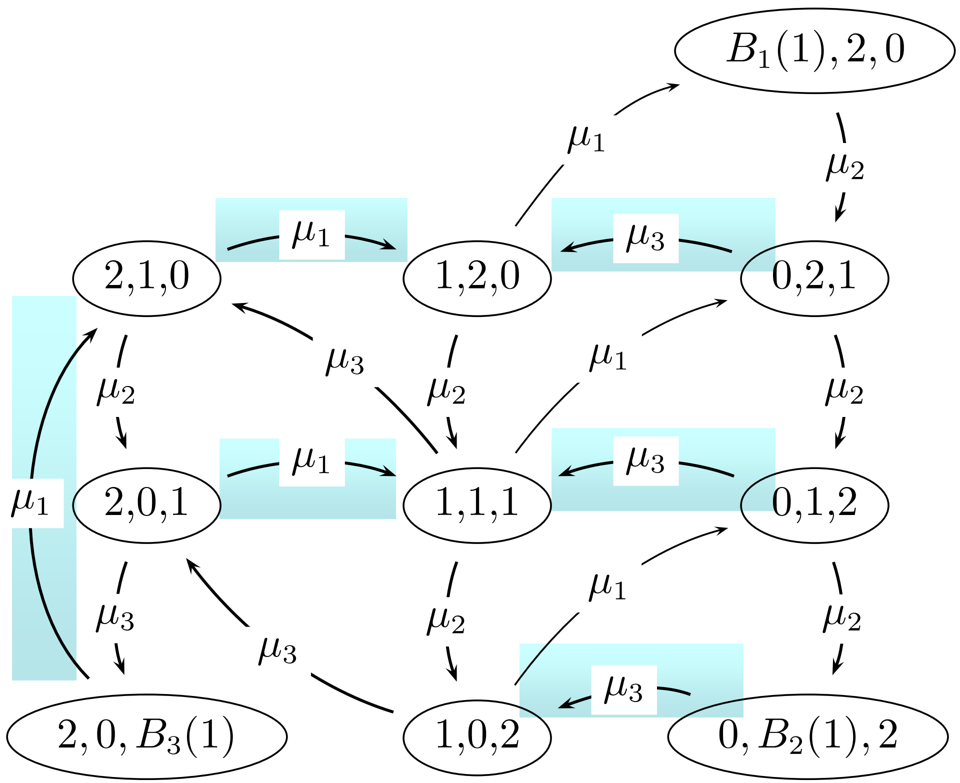


# State transition diagram for the Inter-departure Time is derived from the original state transition diagram

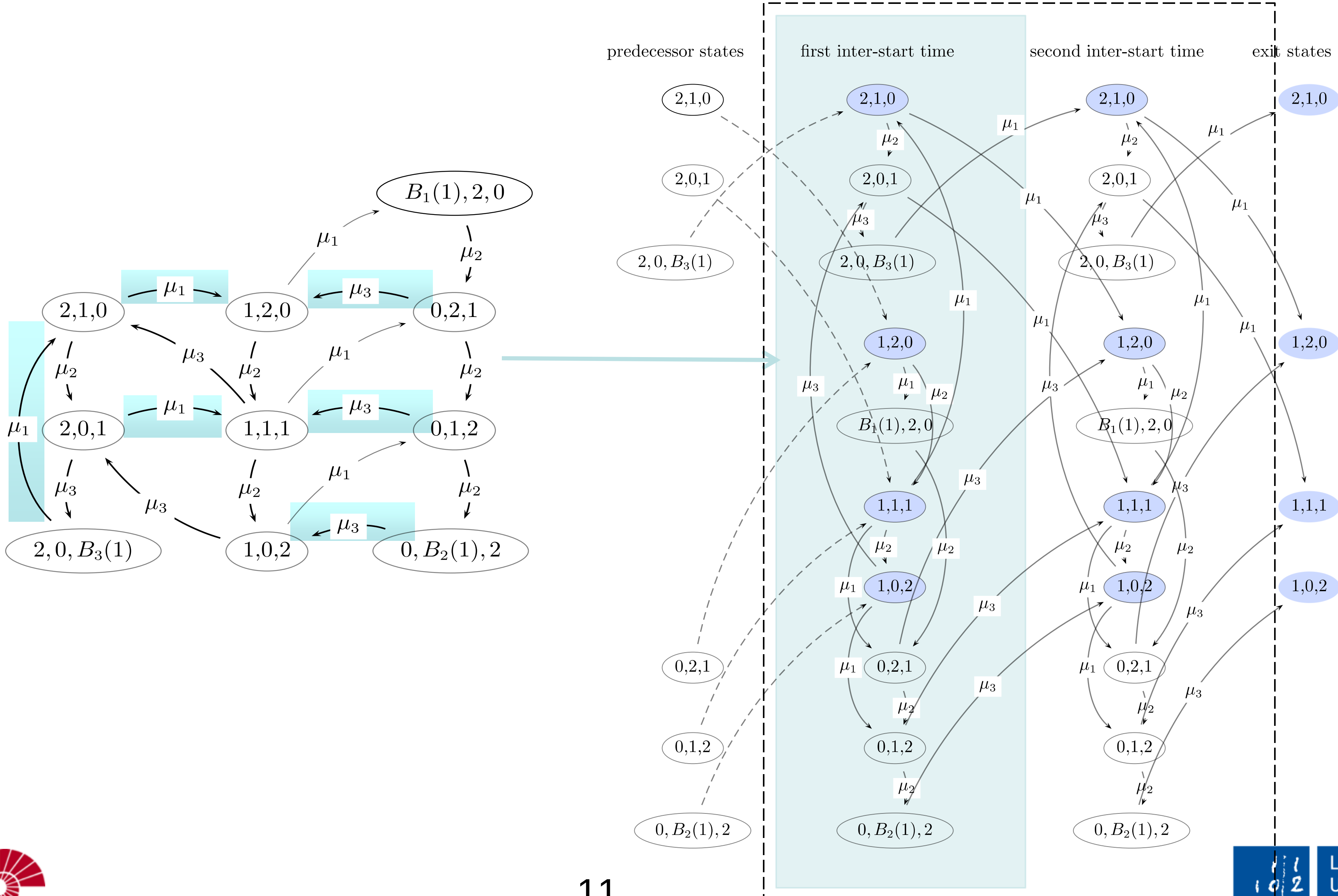


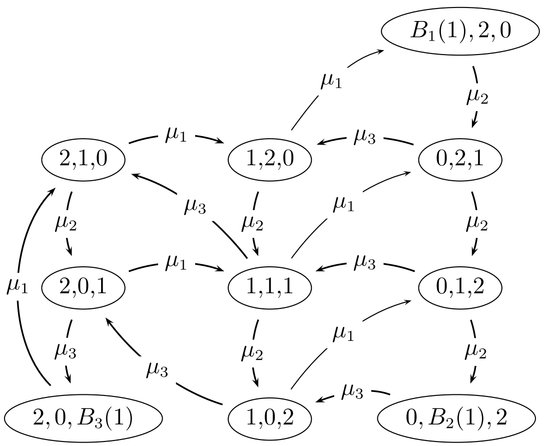
# Transitions with the event "Processing start at station 1"

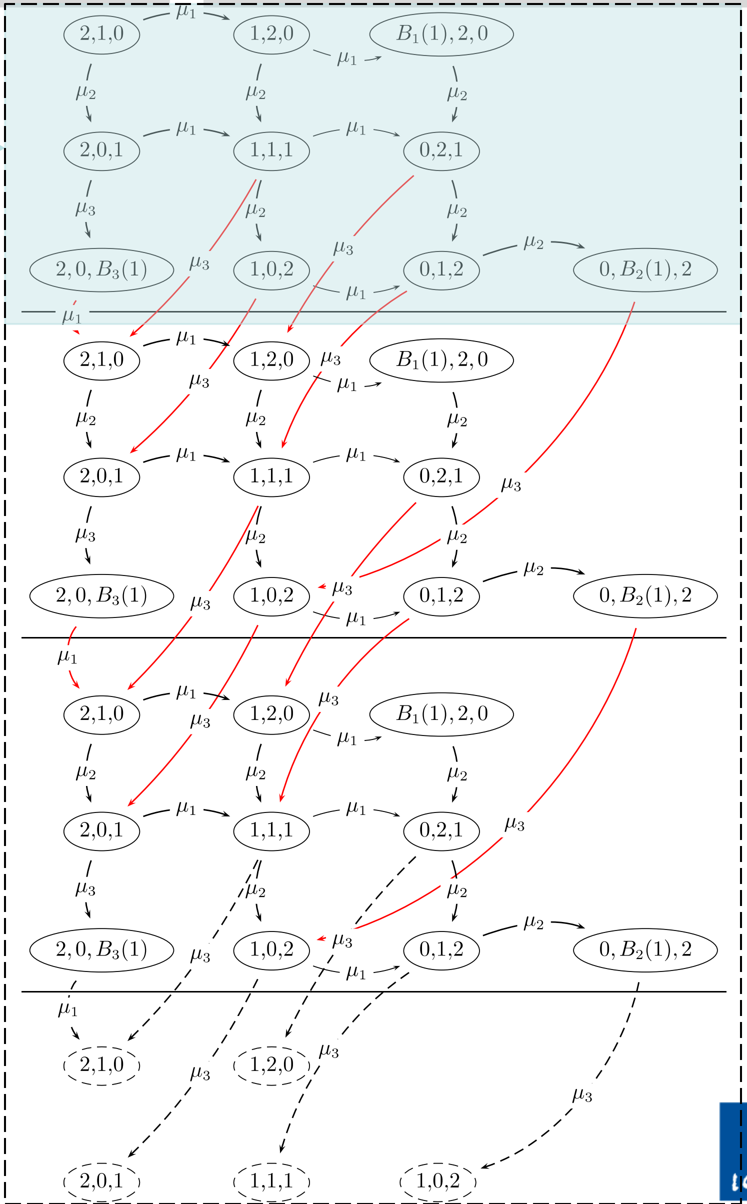


# State transition diagram for the Inter-start Time is derived from the original state transition diagram



# State transition diagram for the Cycle Time is derived from the original state transition diagram



# Plan

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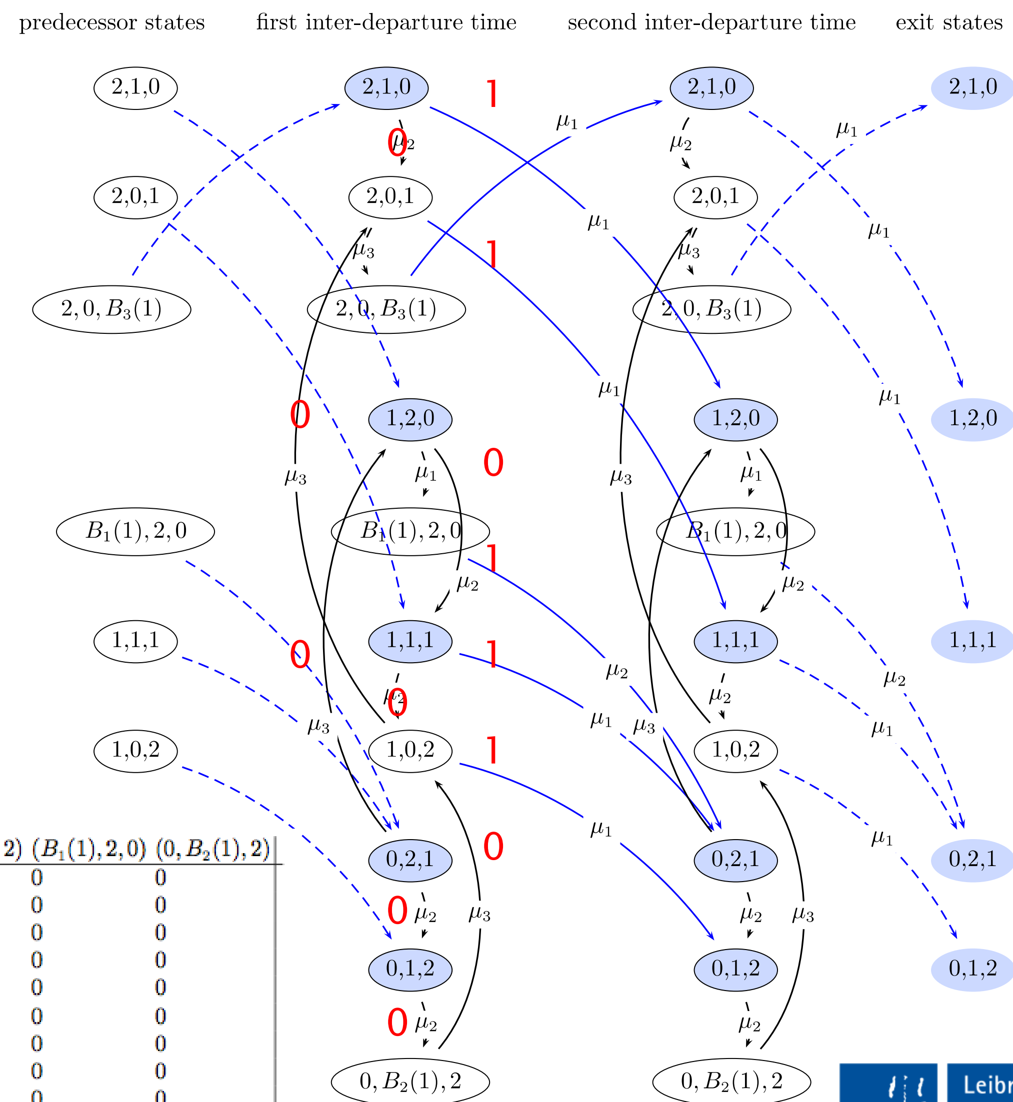
Numerical Results

# Indicator Matrix $G_e$ is formed by using the State Transtion Diagram

Indicator matrix $G_e = \{g^e_{i,j}\}$ with $g_{i,j} = 1$ if event e takes place by a transition from state $i$ to state $j$.

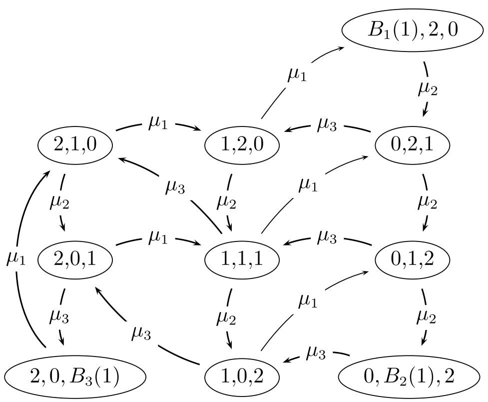
| | $(2,1,0)$ | $(2,0,1)$ | $(2,0,B_3(1))$ | $(1,2,0)$ | $(1,1,1)$ | $(1,0,2)$ | $(0,2,1)$ | $(0,1,2)$ | $(B_1(1),2,0)$ | $(0,B_2(1),2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(2,1,0)$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,0,1)$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $(2,0,B_3(1))$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,2,0)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,1,1)$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $(1,0,2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $(0,2,1)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(0,1,2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(B_1(1),2,0)$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $(0,B_2(1),2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |



# The proposed method yields the exact inter-event time distributions directly by using the system description

System description
(number of stations,
buffer capacities, processing
time distributions, number of
rotating parts,…)

$$Q \qquad G_e$$



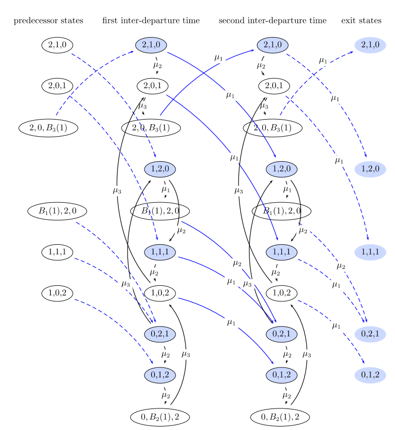

$$F_{T_e}(t) = 1 - \pi_e^{\text{entry}} e^{Q_e t} u,$$

$$\pi_e^{\text{entry}} \left(I + Q_e^{-1} R_e\right) = 0,$$

$$\pi_e^{\text{exit}} = -\pi_e^{\text{entry}} Q_e^{-1} R_e,$$

$$Q_e = Q - R_e.$$

$$R_e = Q \circ G_e$$



# Inter-event Time Distribution is the First Passage Time Distribution when the process starts at the entry states with the corresponding steady-state probabilities

$$F_{T_e}(t) = 1 - \pi_e^{\text{entry}} e^{Q_e t} u,$$

$$f_{T_e}(t) = \frac{d}{dt} F_{T_e}(t) = -\pi_e^{\text{entry}} Q_e e^{Q_e t} u,$$

$$E[T_e] = -\pi_e^{\text{entry}} Q_e^{-1} u$$

$$Var[T_e] = 2\pi_e^{\text{entry}} Q_e^{-2} u - \left(\pi_e^{\text{entry}} Q_e^{-1} u\right)^2$$

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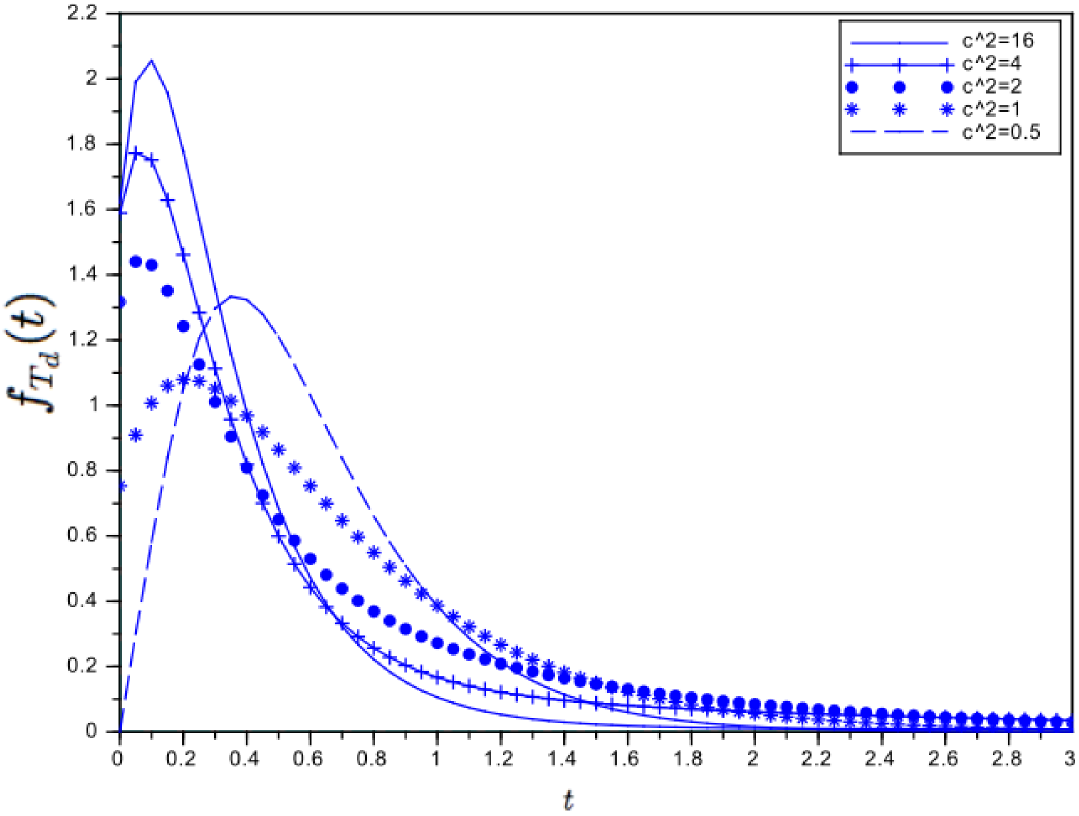
Numerical Results

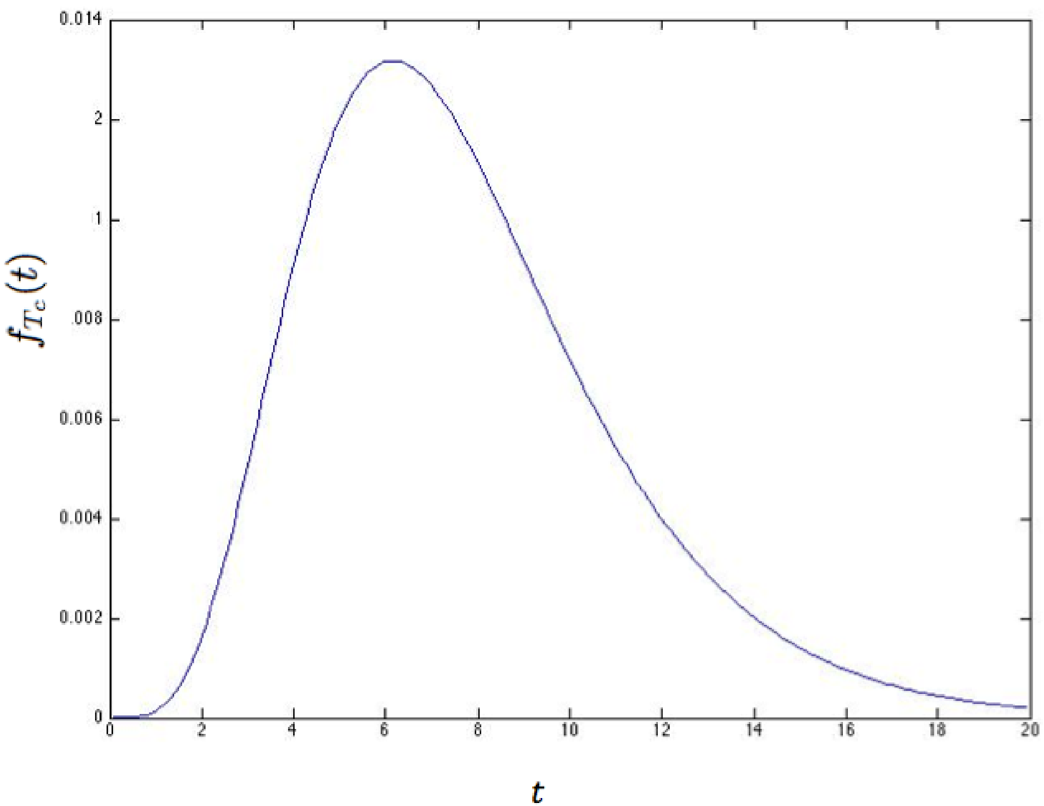
# Probability Density Functions

Figure 1: Probability density functions of the inter-departure time of Station 1 of a closed three-station system with different coefficient of variations of the processing time, $\mu_1 = 3$, $\mu_2 = 2$, $\mu_3 = 2$, $b_i = 2\ \forall i$



# Probability Density Functions

Figure 2: Probability Density Function of the cycle time of Station 1 of a closed three-station production line with exponential servers and 3 pallets, $\mu_1 = 0.7$, $\mu_2 = 0.5$, $\mu_3 = 0.9$

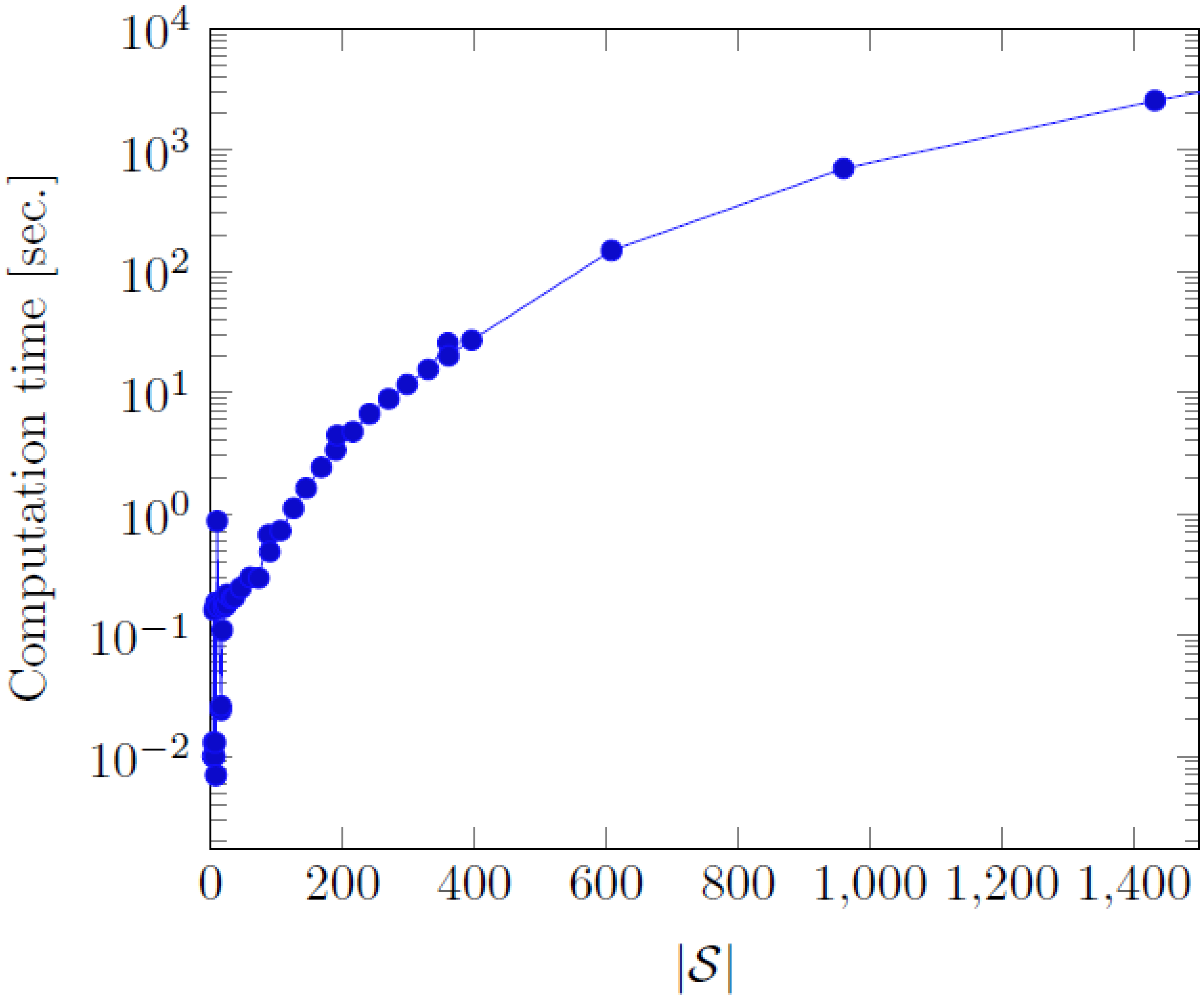


# Numerical Results

Table 2: Computational Results a Four-Station System with a Cox-2 Distribution ($c_i^2 = 2\ \forall i$) and $\mu_1 = 0.7,\ \mu_2 = 0.5,\ \mu_3 = 0,9,\ \mu_4 = 0,7$ and $b_i = 5$ buffers at each station $i = 1, 2, 3, 4$ with an increasing number of customers

| $K$ | $b_i \forall i$ | $M$ | $\lvert\mathcal{S}\rvert$ | $E[T_d]$ | $Var[T_d]$ | | | | $Var[T_s]$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 1 | 8 | 5.9683 | 37.2648 | 37.2648 | 37.2648 | 37.2648 | 37.2648 | 37.2648 | 37.2648 | 37.2648 |
| 4 | 5 | 2 | 32 | 4.3803 | 34.3482 | 34.5416 | 34.4529 | 34.3982 | 32.2664 | 31.6890 | 32.8955 | 32.3160 |
| 4 | 5 | 3 | 88 | 3.7776 | 31.3276 | 31.6701 | 31.5184 | 31.4193 | 29.7086 | 29.5865 | 30.2694 | 29.7999 |
| 4 | 5 | 4 | 192 | 3.4361 | 28.9878 | 29.4527 | 29.2575 | 29.1176 | 27.6203 | 27.7891 | 28.1758 | 27.7498 |
| 4 | 5 | 5 | 360 | 3.2070 | 27.1340 | 27.7063 | 27.4785 | 27.2987 | 25.9291 | 26.3163 | 26.5050 | 26.0939 |
| 4 | 5 | 6 | 608 | 3.0389 | 25.6271 | 26.2968 | 26.0419 | 25.8230 | 24.5383 | 25.1036 | 25.1458 | 24.7349 |
| 4 | 5 | 7 | 960 | 2.9085 | 22.3227 | 24.5705 | 23.8026 | 23.5455 | 23.3773 | 24.0941 | 24.0201 | 23.6016 |
| 4 | 5 | 8 | 1,432 | 2.8179 | 21.9568 | 23.7758 | 23.0949 | 22.6959 | 21.4396 | 23.0637 | 22.5047 | 22.1247 |
| 4 | 5 | 9 | 2,000 | 2.7516 | 21.5912 | 23.1401 | 22.5089 | 21.9523 | 21.2743 | 22.5346 | 22.0022 | 21.5877 |
| 4 | 5 | 10 | 2,616 | 2.7023 | 21.2731 | 22.6320 | 22.0235 | 21.2993 | 21.0887 | 22.1055 | 21.5780 | 21.1241 |
| 4 | 5 | 11 | 3.232 | 2.6655 | 21.0196 | 22.2288 | 21.6248 | 20.7200 | 20.9321 | 21.7650 | 21.2282 | 20.7323 |
| 4 | 5 | 12 | 3.800 | 2.6387 | 20.8287 | 21.9123 | 21.2988 | 20.2585 | 20.8197 | 21.5026 | 20.9484 | 20.4050 |

# Computation Time



# Conclusions

A method to determine the exact distributions of the Inter-departure, Inter-start and cycle Time for CQN with phase-type distributed processing times and blocking

Exact results for systems with 10 exponential stations and 5 Cox-2 stations are given

Exact results can be used to test the accuracy of approximation methods

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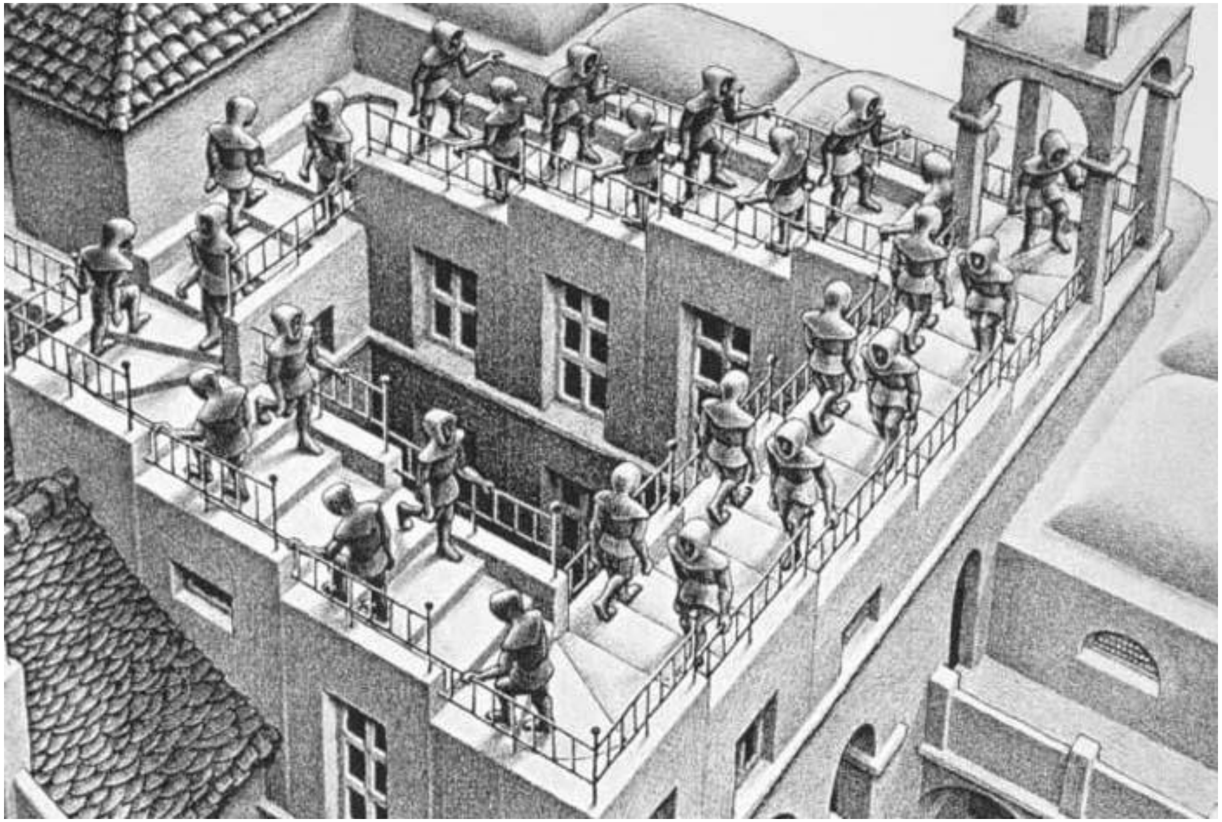



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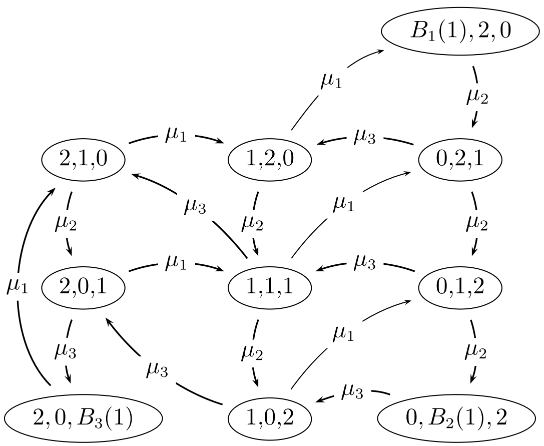
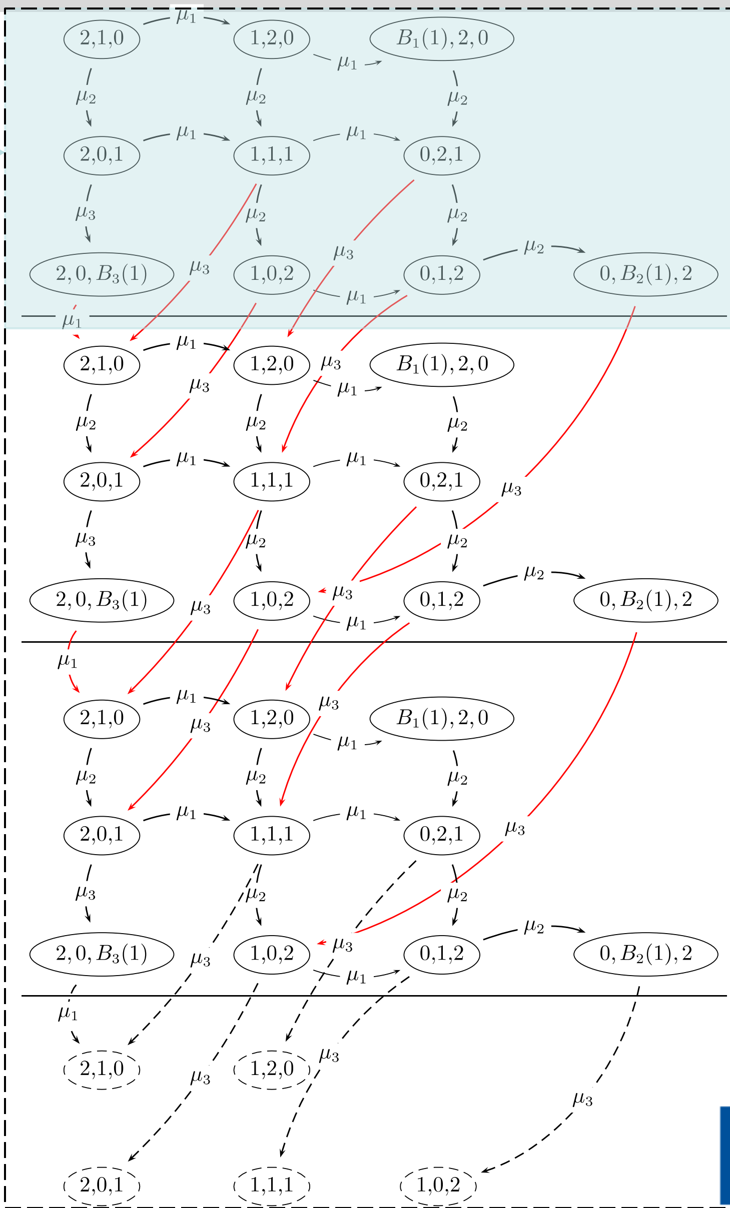
# Numerical Results

Table 1: Computational Results of a Three-Station System with an Exponential Distribution ($c_i^2 = 1\ \forall i$) and $\mu_1 = 0.7,\ \mu_2 = 0.5,\ \mu_3 = 0,9$

| $K$ | $b_i \forall i$ | $M$ | $\lvert\mathcal{S}\rvert$ | $E[T_d]$ | $Var[T_d]$ | | | $Var[T_s]$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 3 | 10 | 2.5966 | 4.0873 | 5.0772 | 4.7236 | 5.5674 | 5.5674 | 5.5674 |
| 3 | 2 | 5 | 18 | 2.2805 | 3.7279 | 4.5293 | 4.1966 | 4.4601 | 4.7100 | 4.4971 |
| 3 | 3 | 6 | 25 | 2.1840 | 3.6016 | 4.3966 | 4.1575 | 4.2448 | 4.4848 | 4.3439 |
| 3 | 4 | 8 | 36 | 2.1092 | 3.4893 | 4.2401 | 3.9848 | 4.1921 | 4.2922 | 4.1596 |
| 3 | 5 | 9 | 46 | 2.0745 | 3.4365 | 4.1755 | 3.9956 | 4.1031 | 4.2029 | 4.1119 |
| 3 | 6 | 11 | 60 | 2.0482 | 3.3958 | 4.1160 | 3.9261 | 4.0876 | 4.1323 | 4.0436 |
| 3 | 7 | 12 | 73 | 2.0334 | 3.3722 | 4.0839 | 3.9505 | 4.0478 | 4.0927 | 4.0309 |
| 3 | 8 | 14 | 90 | 2.0226 | 3.3555 | 4.0577 | 3.9193 | 4.0420 | 4.0629 | 4.0017 |
| 3 | 9 | 15 | 106 | 2.0158 | 3.3445 | 4.0415 | 3.9442 | 4.0234 | 4.0443 | 4.0014 |
| 3 | 10 | 17 | 126 | 2.0109 | 3.3371 | 4.0291 | 3.9292 | 4.0208 | 4.0307 | 3.9879 |
| 3 | 11 | 18 | 145 | 2.0077 | 3.3318 | 4.0209 | 3.9505 | 4.0118 | 4.0218 | 3.9916 |
| 3 | 12 | 20 | 168 | 2.0054 | 3.3283 | 4.0148 | 3.9432 | 4.0105 | 4.0153 | 3.9851 |
| 3 | 13 | 21 | 190 | 2.0038 | 3.3257 | 4.0106 | 3.9600 | 4.0060 | 4.0109 | 3.9895 |
| 3 | 14 | 23 | 216 | 2.0027 | 3.3240 | 4.0075 | 3.9563 | 4.0053 | 4.0077 | 3.9863 |
| 3 | 15 | 24 | 241 | 2.0019 | 3.3227 | 4.0054 | 3.9692 | 4.0031 | 4.0055 | 3.9903 |
| 3 | 16 | 26 | 270 | 2.0014 | 3.3219 | 4.0038 | 3.9673 | 4.0027 | 4.0039 | 3.9887 |
| 3 | 17 | 27 | 298 | 2.0010 | 3.3212 | 4.0027 | 3.9769 | 4.0016 | 4.0028 | 3.9920 |
| 3 | 18 | 29 | 330 | 2.0007 | 3.3208 | 4.0020 | 3.9759 | 4.0014 | 4.0020 | 3.9912 |
| 3 | 19 | 30 | 361 | 2.0005 | 3.3205 | 4.0014 | 3.9829 | 4.0008 | 4.0014 | 3.9938 |
| 3 | 20 | 32 | 396 | 2.0004 | 3.3203 | 4.0010 | 3.9825 | 4.0007 | 4.0010 | 3.9934 |

# State transition diagram for the Cycle Time is derived from the original state transition diagram



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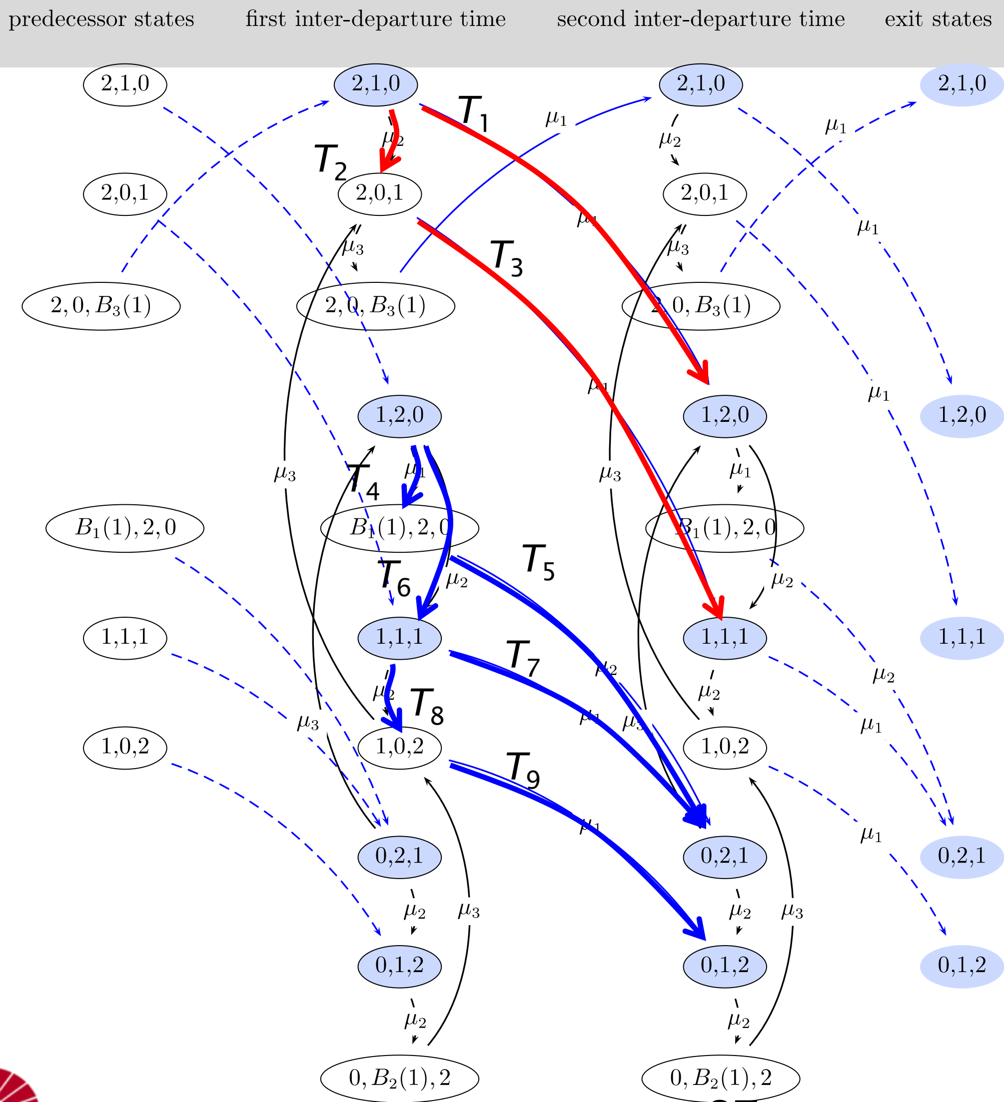
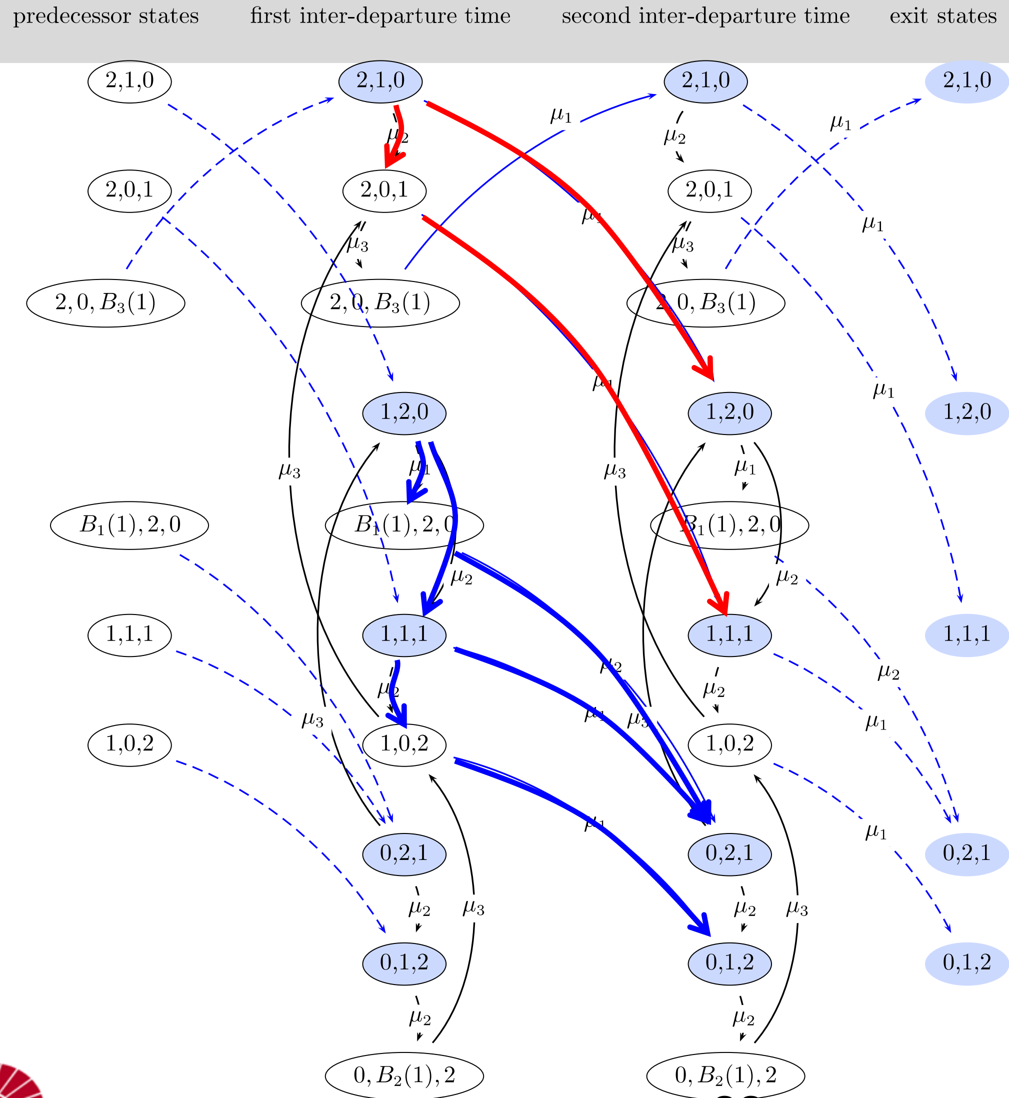
# Inter-event Time is the First Passage Time between an entry state and an exit state of the extended state space




Inter-departure time from state (2,1,0): Min{$T_1$, $T_2$+$T_3$}

Inter-departure time from state (1,2,0): Min{$T_4$ + $T_5$,
$T_6$+$T_7$,
$T_6$+$T_8$+$T_9$}

# Inter-event Time Distribution is the First Passage Time Distribution when the process starts at the entry states with the corresponding steady-state probabilities




$$F_{T_e}(t) = 1 - \pi_e^{\text{entry}} e^{Q_e t} u,$$

$$f_{T_e}(t) = \frac{d}{dt} F_{T_e}(t) = -\pi_e^{\text{entry}} Q_e e^{Q_e t} u,$$

$$E[T_e] = -\pi_e^{\text{entry}} Q_e^{-1} u$$

$$Var[T_e] = 2\pi_e^{\text{entry}} Q_e^{-2} u - \left(\pi_e^{\text{entry}} Q_e^{-1} u\right)^2$$



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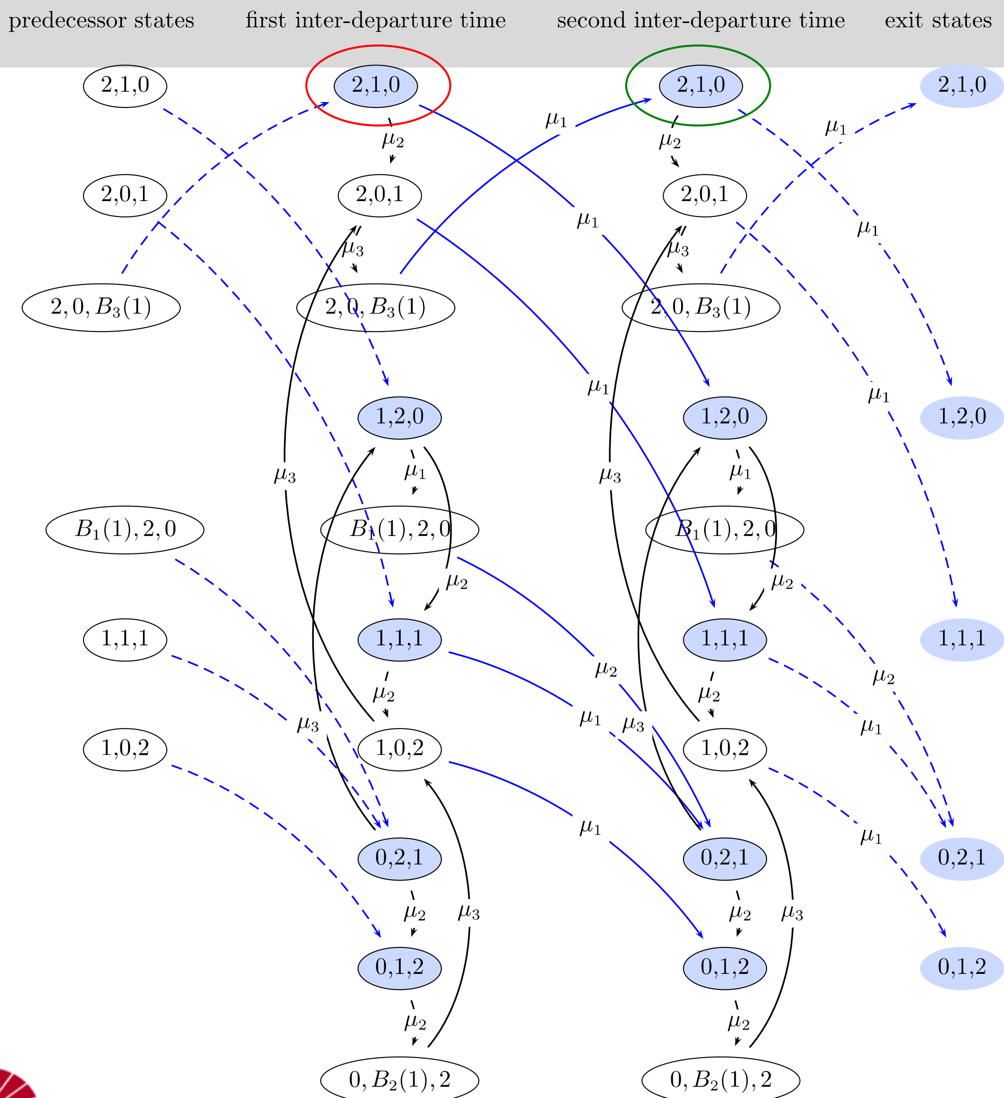
# Steady-state entry probability distribution is the same as the steady-state exit probability distribution




entry states

exit states

$$F_{T_e}(t) = 1 - \pi_e^{\text{entry}} e^{Q_e t} u,$$

$$f_{T_e}(t) = \frac{d}{dt}F_{T_e}(t) = -\pi_e^{\text{entry}} Q_e e^{Q_e t} u,$$

$$\pi_e^{\text{exit}} = -\pi_e^{\text{entry}} Q_e^{-1} R_e,$$

$$\pi_e^{\text{entry}} = \pi_e^{\text{exit}},$$

$$\pi_e^{\text{entry}} \left(I + Q_e^{-1} R_e\right) = 0,$$

$$\pi_e^{\text{entry}} u = 1.$$