Optimal Control of a Multi-state Manufacturing System: Control of Production Rate and Temporary Increase in Capacity *

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Abstract

In planning and scheduling of production systems manufacturers have two main strategies for responding to uncertainty: they build inventory to hedge against periods in which the production capacity is not sufficient to satisfy demand, or they temporarily increase the production capacity. We consider the problem of minimizing the long-run average cost of holding inventory and/or purchasing extra capacity for a single facility producing a single part-type. The optimal production policy is of a *hedging point policy* type where two hedging levels are associated with each discrete capacity state of the system: a positive hedging level (inventory target) and a negative one (backlog level below which extra capacity should be purchased). We establish some ordering of the hedging levels, derive equations satisfied by the steady-state probability distribution of the inventory/backlog, and describe a program for specifying the optimal hedging levels.

1 Introduction

In planning and scheduling of production systems manufacturers have two main strategies for responding to uncertainty: they build inventory to hedge against periods in which the production capacity is not sufficient to satisfy demand (high demand or low capacity), or they temporarily increase the production capacity (for example, by hiring temporary workers, adding overtime and extra shifts, or by outsourcing part of the production). Each of these options has an associated cost and the task of a production planner and scheduler is to strike a balance between them.

This problem can be formulated in the framework of flow control of manufacturing systems where inventory/backlog of discrete parts is approximated by continuous variables and the dynamics of the inventory/backlog by differential equations; changes in demand and production capacity are modeled by a discrete state stochastic process, often a Markov chain. (For a detailed review and a

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comprehensive list of references on flow control of manufacturing systems, see two recent books by Gershwin [1] and Sethi and Zhang [10].)

We consider the case of a single product and a single production facility (single machine or multiple parallel machines). In this case the structure of the optimal production policy has been established (see, e.g., Sethi et al. [9]; Sethi et al. do not directly consider this problem; however, their results, with some modifications, imply the structure of the controller in our case). The optimal production policy is of a *hedging point* type where two hedging levels are associated with each discrete state of the system: a positive hedging level (inventory target) and a negative one (backlog level below which extra capacity should be purchased). Given this structure, the problem of specifying the optimal production control policy becomes that of determining the optimal hedging points. This is the problem we address in this paper. We establish some ordering of the hedging levels, derive equations satisfied by the steady-state probability distribution of the inventory/backlog, and describe a program for specifying the optimal hedging levels.

A similar problem without the possibility of purchasing extra capacity is addressed by Sharifnia [11]. Hu and Dong [3], [4], and Liberopoulos and Hu [7] derive similar order relationship to what is established in this paper between hedging levels for the case where purchase of extra capacity is not considered. Some of the results of Sethi et al. [8] are along the same lines. Purchase of extra capacity is considered in Hu [2] in an indirect way. A more restricted case of the problem considered in this paper is analyzed in Huang, Hu and Vakili [5] and the optimum hedging levels are derived analytically.

The rest of the paper is organized as follows. Problem formulation and preliminaries are given in Section 2; Structural properties of the hedging levels are derived in Section 3, and the equations satisfied by the steady-state distribution are derived in Section 4.

2 Problem formulation and preliminaries

We assume that inventory/backlog evolves continuously and that changes in production capacity or the demand rate are due to discrete events. Let $\alpha = \{\alpha(t); t \geq 0\}$ denote a stochastic process representing random changes in the production capacity and/or random fluctuations in the demand rate. Let $\mathbf{x} = \{x(t); t > 0\}$ denote the inventory/backlog process. We assume that only one part type is produced, so $x(t)$ is a scalar. $(\mathbf{x}, \alpha) = \{(x(t), \alpha(t)); t \geq 0\}$ is the state process of the system.

The production rate and the rate of buying extra capacity at time t are denoted, respectively, by $u(x, \alpha)(t)$ and $v(x, \alpha)(t)$. At any time t there is a default production capacity that depends on α and is denoted by $C_\alpha(t)$; moreover, it is possible to purchase extra capacity. The amount of purchase is limited to a set denoted by C. The dynamics of $x(t)$ is given by

$$\frac{dx}{dt}(t) = u(x,\alpha)(t) + v(x,\alpha)(t) - d_\alpha(t). \tag{1}$$

We assume that α is a time-homogeneous irreducible Markov chain on the finite set $\{1,\cdots,m\}$; let $Q = (q_{ij})$ denote the infinitesimal generator of α. We assume that α is not affected by the production control policy, i.e., Q is not a function of u, or v.

Let $g(x)$ be the cost per unit time of holding x units of inventory or backlog, and let $h(v)$ be the cost per unit time of buying extra capacity at rate v. Let π denote a production policy belonging to an appropriately defined set of control policies Π. We define the cost associated with π to be the long-run average cost:

$$J^\pi = \lim_{T\to\infty} \frac{1}{T} E[\int_0^T (g(x) + h(v))dt | x(0), \alpha(0)]. \tag{2}$$

An optimal control policy in this setting is a feasible control policy $\pi^* \in \Pi$ that satisfies

$$J^{\pi^*} \leq J^\pi \quad \text{for all } \pi \in \Pi. \tag{3}$$

We make the following additional assumptions:

- $C_i = [0, r_i]$; $i = 1, \cdots, m$;
- $C = [0, \bar{r}]$; $\bar{r} + r_i > d_i$ for all $i \in \{1, \cdots, m\}$, i.e., in all states it is possible to satisfy demand using a mixture of default capacity and extra capacity, if needed;
- $g(x) = c^+x^+ + c^-x^-$ where $x^+ = \max(x,0)$, $x^- = \max(-x,0)$, i.e., the inventory/backlog cost is piecewise linear;
- $h(v) = c_v v$, i.e., the cost of purchasing extra capacity is linear in the rate of purchase.

Let π^*, J^*, and V_i^* denote, respectively, the optimal policy, the optimal cost, and the optimal value function. The HJB equation in this context is

$$\begin{aligned} \min_{u,v}\{\frac{dV_i^*(x)}{dx}(u+v-d) + h(v)\} = \\ -\sum_j q_{ij}V_j^*(x) - g(x) + J^*. \end{aligned} \tag{4}$$

A *hedging point policy* is characterized by m vectors of parameters $\mathbf{z}_i = (z_i, z_i^{'})$ where $z_i \geq z_i^{'}$ $(i = 1, 2, \cdots, m)$. Given $\mathbf{z} = (\mathbf{z}_1, \cdots, \mathbf{z}_m)$, the policy is defined as follows:

$$\pi^{\mathbf{z}}(x,i) = \begin{cases} (0,0) & x > z_i \\ (\min(r_i, d_i), 0) & x = z_i \\ (r_i, 0) & z_i^{'} < x < z_i \\ (r_i, \max(0, d_i - r_i)) & x = z_i^{'} \\ (r_i, \bar{r}) & x < z_i^{'} \end{cases} \tag{5}$$

In words, assume that the discrete state of the system is i, i.e., $\alpha = i$, then if the inventory is larger that z_i production is stopped and the inventory is allowed to deplete until the level z_i is reached; at z_i, the inventory is maintained at this level if i is a feasible state ($r_i > d_i$) or the maximum default capacity is used to prevent rapid depletion of inventory below z_i. For values of inventory below z_i and above z_i' the maximum default capacity is used. For infeasible states, once the backlog reaches the level z_i', sufficient extra capacity is purchased to keep the level of backlog at z_i'. When the backlog level is below z_i', the maximum extra capacity is purchased to reduce the amount of backlog as rapidly as possible.

3 The order of optimal hedging points

From results of [9] it follows that

Theorem 1 *V_i^* is a convex function; the optimal control policy is a hedging point policy; z_i^* is the minimum of $V_i^*(x)$ and $z_i^{*\prime}$ is the minimum of $V_i^*(x) + c_v x$.*

In addition, using a similar argument as that used in [3], it can be shown that

Lemma 1 *$z_i^* \geq x^* \geq z_i^{*\prime}$, $i = 0, 1, 2, ...m$, where x^* is the minimum of convex cost function $g(x)$; $x^* = 0$ in our case.*

Our main results on the order of hedging points are (their proofs can be found in [6])

Proposition 1 *Let i be a feasible state, i.e., $r_i > d_i$; then*

$$z_i^* \leq \max\{z_j^* : j \neq i; q_{ij} \neq 0\}$$

Proposition 2 *Let i be an infeasible state, i.e., $r_i < d_i$; then*

$$\min\{z_j^{*\prime} : j \neq i; q_{ij} \neq 0\} \leq z_i^{*\prime}$$

We now consider the above results in some special cases of practical interest. First, consider the system with m identical failure prone machines. The capacity state of this system forms a birth-death process on $\{0, \cdots, m\}$. Consider a fixed demand rate and assume that when l or smaller number machines are up it is not possible to satisfy demand, whereas the rate of production when $l+1$ or larger number of machines are up is higher than the demand rate. In other words, all states in $\{0, \cdots, l\}$ are infeasible while all states in $\{l+1, \cdots, m\}$ are feasible. The next proposition establishes the order of the hedging points in this case and similar cases.

Proposition 3 *Assume that the capacity process α is a birth-death process on $\{0, \cdots, m\}$. Also assume that the states $\{0, \cdots, l\}$ are infeasible and the states $\{l+1, \cdots, m\}$ are feasible. Then*

$$z_l^{*\prime} \leq \cdots \leq z_0^{*\prime} \leq x^* \leq z_m^* \leq \cdots \leq z_{l+1}^*.$$

Next consider m failure prone machines that are not necessarily identical. The capacity state of this system can be represented by an m-dimensional vector of ones and zeros. A one at location i of the vector denotes that machine i is up and a zero denotes that it is down. Let C_m^k represent the set of all states where m machines are up. Note that in the Markov chain of the capacity process a transition is always from C_m^k to C_m^{k-1} or to C_m^{k+1}. Let feasible(C_m^k) be the set of all feasible states in C_m^k and infeasible(C_m^k) the set of all infeasible states. Furthermore, let

$$z_{C_m^k}^* = \max\{z_{C_m^k}^*(i) : i \in \text{feasible}(C_m^k)\};$$

and

$$z_{C_m^k}^{*\prime} = \min\{z_{C_m^k}^{*\prime}(i) : i \in \text{infeasible}(C_m^k)\}.$$

Then, it can be shown ([6]) that

Proposition 4 *Let l be the largest number for which the set infeasible(C_m^l) is non-empty; then*

$$z_{C_m^l}^{*\prime} \leq z_{C_m^{l-1}}^{*\prime} \leq \cdots \leq z_{C_m^0}^{*\prime} \leq x^*.$$

Let n be the smallest number for which the set infeasible(C_m^n) is nonempty; then

$$x^* \leq z_{C_m^m}^* \leq z_{C_m^{m-1}}^* \leq \cdots \leq z_{C_m^n}^*.$$

4 Optimal hedging points

One way to determine the optimal hedging points is to obtain the total cost function as a function of a set of tentative hedging points. Finding the optimizer of this function yields the optimum hedging points. The total cost is the infinite horizon average cost of the policy, or, equivalently, the expected cost with respect to the steady state distribution of the state process. We describe a method for evaluating the steady state distribution.

4.1 Probability flow balance equations

Assume that there are l infeasible and $m+1-l$ feasible capacity states. Using the results of the previous section, we assume the following order between the hedging points (rename the cpacity states if necessary).

$$z_l' \leq \cdots \leq z_0' \leq 0 \leq z_m \leq \cdots \leq z_{l+1}.$$

Let $R^i, R^{i'}, S, T$ denote the following regions:
$R^i = \{x : z_i < x < z_{i+1}\} \quad i = l, \cdots, m;$
$S = \{x : z_0' < x < z_m\};$
$R^{i'} = \{x : z_i' < x < z_{i-1}'\} \quad i = 1, \cdots, l;$
$T = \{x : x < z_l'\}.$
Define the probability distribution function by

$$P^t(x, j) = prob\{X(t) < x, \alpha(t) = j\}$$

and the instantaneous net production rate by

$$\Phi_i^j = u(x, j) - d,$$

$x \in R^i, R^{i'}, S, T$, $j = 0, 1, \cdots, m$. Then, we have

$$\begin{aligned} P^{t+h}(x, k) = \sum_{j \neq k} q_{jk} h P^t(x - \Phi_i^j h, j) + \\ (1 + q_{kk} h) P^t(x - \Phi_i^j h, k) + o(h). \end{aligned}$$

Assume that the steady state probability distribution exist, i.e., $P(x.k) = \lim_{t \to \infty} P^t(x, k)$, $k = 0, \cdots, m$. Then by rearranging and letting $t \to \infty$ and $h \to 0$, we have

$$\Phi_i^j \frac{dP(x, k)}{dx} = \sum_{j=1}^{m} q_{jk} P(x, j),$$

Or, in matrix form

$$\frac{dP(x)}{dx} = P(x) A_i$$

where $x \in R^i$, $R^{i'}$, S, or T, $A_i = Q\Omega_i^{-1}$, $\Omega_i = diag[\Phi_i^0, \cdots, \Phi_i^m]$, and $P(x) = [P(x, 0), \cdots, P(x, m)]$.

Note that the steady state probability distribution function is left continuous and its jumps are only at the hedging points that have nonzero probability mass.

4.2 Boundary conditions

Let the probability mass at z_i and z_i' be denoted by $p(z_i) = prob\{x = z_i\}$, $p(z_i') = prob\{x = z_i'\}$ for $i = 0, 1, 2, \cdots, m$. Some of them would be zero if z_i's or z_i''s are infinite or transient. also let

$$\begin{aligned} P^+(z_i) &= \lim_{\delta \to 0^+} P(z_i + \delta), \\ P^-(z_i) &= \lim_{\delta \to 0^-} P(z_i + \delta), \\ P^+(z_i') &= \lim_{\delta \to 0^+} P(z_i' + \delta), \\ P^-(z_i') &= \lim_{\delta \to 0^-} P(z_i' + \delta). \end{aligned}$$

We have our first set of boundary conditions:

$$P(z_i) = P^-(z_i) = P^+(z_i) - p(z_i)e_i,$$
$$P(z_i^{'}) = P^-(z_i^{'}) = P^+(z_i^{'}) - p(z_i^{'})e_i.$$

where e_i is the row vector with $m+1$ zero entries except with 1 at ith entry.

The second set of boundary conditions are

$$P^-(z_l') = 0, P^+(z_i) = \pi_i,$$

where π_i is the steady state probability of the capacity process being in state i and can be calculated from $\pi Q = 0$ and $\sum \pi_i = 1$.

The number of unknowns is the number of probability masses of hedging points. According to our control policy, the number of non-transient positive hedging points is equal to the number of feasible states and the number of non-transient negative hedging points is equal to the number of infeasible states; therefore, the number of all non-transient hedging points is equal to the total number of states of production capacity, i.e., the number of unknown probability masses is equal to the number of states of production capacity.

It can be shown that the number of boundary conditions is equal to the number of states of production capacity. Also the number of equations is equal to the number of states of production capacity. Since we use the $x \to z_l^{'}$ instead of $x \to \infty$ for our boundary condition, the sign of eigenvalues does not count anymore in our set of probability distribution equation, that is, all equations can be used to solve for the unknowns in our case. Hence, we have exactly the right number of equations to solve for all unknowns without imposing any special structure on the state process of production capacity.

4.3 Computation of optimal hedging points

Propositions 1 and 2 give sets of relationships on the order of hedging points for a general state process. These conditions correspond to a subset of the $\mathbf{z}$ space. Once a point in this subset is selected the steady state probability distribution and hence the average cost corresponding to the policy $\pi^{\mathbf{z}}$ can be calculated as

$$\begin{aligned} J(\mathbf{z}) &= J(z_0', \cdots, z_l', z_{l+1}, \cdots, z_m) \\ &= \int_{z_l'}^{z_{l+1}} g(x) \sum_{\alpha} \pi_\alpha dP(x, \alpha). \end{aligned}$$

The problem of determining the optimal hedging point policy then turns into a constrained nonlinear optimization problem. The cost function $J(\mathbf{z})$ is nonlinear and the constraints are linear with simple inequality structure.

References

[1] S. Gershwin, *Manufacturing Systems Engineering,* Prentice Hall, 1994.

[2] J.Q. Hu, "Production Rate Control for Failure Prone Production With No Backlog Permitted," *IEEE Transactions on Automatic Control,* Vol.40, No.2, pp. 291-295, 1995.

[3] J.Q. Hu and D. Xiang, "Structure Properties of Optimal Production Controllers of Failure Prone Manufacturing Systems," *IEEE Transactions on Automatic Control,* Vol.39, No.3, pp. 640-643, 1994.

[4] J.Q. Hu and D. Xiang, "Monotonicity of Optimal Flow Control for Failure Prone Production Systems," *Journal of Optimization Theory and Applications,* Vol. 86, No.1, pp. 57-71, July 1995.

[5] L. Huang, J.Q. Hu, P. Vakili, "Optimal control of failure prone production system with the possibility of using extra capacity", in "Recent Advances in Manufacuturing Engineering" G. Yin and Q. Zhang Eds. pp. 31-60, Springer-Verlag. 1996.

[6] L. Huang, J.Q. Hu, P. Vakili, "Optimal Control of a Multi-state Manufacturing System: Control of Production Rate and Temporary Increase in Capacity," *Proceedings of the 37th IEEE Conference on Decision and Control,* pp. 2155-2130, 1998.

[7] G. Liberopoupos and J.Q. Hu, "On the Order of Optimal Hedging Points in a Class of Manufacturing Flow Control Models," *IEEE Transactions on Automatic Control,* Vol.40, No.2, pp. 282-286, 1995.

[8] S. Sethi, H.M. Soner, Q. Zhang, and J. Jiang, "Turnpike Sets and their Analysis in Stochastic Production Planning Problems," *Math. OR,* Vol. 17, pp. 932–950, 1992.

[9] S.P. Sethi, W. Suo, M. Taksar, and Q. Zhang, "Optimal Production Planning in a Stochastic Manufacturing System with Long-Run Average Cost," *JOTA*, 1995.

[10] S.P. Sethi and Q. Zhang, *Hierarchical Decision Making in Stochastic Manufacturing Systems,* Birkhauser, Boston, 1994.

[11] A. Sharifnia, "Optimal Production Control of a Manufacturing System with Machine Failures," *IEEE Transactions on Automatic Control,* Vol. AC-33, No. 7, pp. 620-625, July 1988.