ON DECOMPOSITION METHODS
FOR THE ANALYSIS OF PRODUCTION LINES
WITH UNRELIABLE MACHINES AND FINITE BUFFERS

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Abstract

We consider production lines consisting of a series of machines separated by finite buffers. The processing time of each machine is deterministic and all the machines have the same processing time. All machines are subject to failures. As usually the case for production systems we assume that the failures are operation dependent [3,7]. Moreover, we assume that the times to failure and the times to repair are exponentially distributed. To analyze such systems, a decomposition method was proposed by Gershwin [11]. The computational efficiency of this method was later significantly improved by the introduction of the so-called DDX algorithm [5,6]. In general, this method provides fairly accurate results. There are however cases for which the accuracy of this decomposition method may not be so good. This is the case when the reliability parameters (mean times to failure and mean times to repair) of the different machines have different orders of magnitude. Such a situation may be encountered in real production lines. In [8] an improvement of Gershwin's original decomposition method has been proposed that in general provides more accurate results in the above mentioned situation. This other method is referred to as the GE-method. The basic difference between the GE method with that of Gershwin is that it uses a two-moment approximation instead of a one-moment approximation of the repair time distributions of the equivalent machines. There are however cases for which the accuracy of the GE-method is not be as good as expected. This is the case for example when the buffer sizes are too small in comparison with the mean times to repair or in the case of closed-loop production lines. We present in this paper a new decomposition method that is a better approximation of the repair time distribution.

Keywords: production lines, unreliable machines, finite buffers, decomposition method, generalized exponential distributions

1. Introduction

In this paper, we consider production lines consisting of a series of machines separated by finite buffers. The processing time of each machine is deterministic, i.e., a fixed amount of time is required to perform the operation. Moreover, we assume that all the machines have the same processing time, that is we restrict our attention to so-called *homogeneous lines* [6]. All machines are subject to failures. As is usually the case for production systems we assume that the failures are operation dependent [3,7]. This means that a machine can fail only while it is working. Moreover, we assume that the times to failure and the times to repair are exponentially distributed. Finally, we assume that there are always raw parts available at the input and that there is always room to accommodate the finished parts at the output. Models of production lines with deterministic processing times and exponentially distributed times to failure and times to repair are referred to as *asynchronous models* [7].

A number of methods have been developed for analyzing production lines with unreliable machines and finite buffers (also called transfer lines). See [7] for a survey and a list of references. Obtaining exact analytical solutions of asynchronous models of production lines is in general not feasible. As a result, different models have been used to approximate the behavior of asynchronous models [7]: the synchronous model [2] and the continuous flow model [16]. These models provide a good approximation of the original asynchronous model as long as the average times to failures are significantly larger than the processing times, which is usually the case in production systems [1]. For both models, exact solutions of a line consisting of two machines separated by a finite buffer can be obtained; see e.g. [3,13] in the case of the synchronous model and [9,11] in the case of the continuous flow model.

The analysis of longer lines is based on approximation methods [7]. Among these methods, the decomposition method proposed by Gershwin [11] in the context of the synchronous model appears to be quite accurate. Moreover, using the iterative algorithm proposed in [5], this method can be made very efficient and reliable. A similar decomposition method was proposed in the case of the continuous flow model [6]. However, there are situations for which the accuracy of this decomposition method may not be so good. This is the case when the reliability parameters (mean times to failure and mean times to repair) of the different machines have different order of magnitudes. Such a situation may be encountered in real production lines.

In [8] an improvement of Gershwin's original decomposition method was proposed that provides accurate results even in the above mentioned situation. This other method is referred to as the GE-method. The basic difference between the GE method with that of Gershwin is that it uses a two-moment approximation instead of a one-moment approximation of the repair time distributions of the equivalent machines. Even though the GE-method is fairly robust, there are still situations for which the accuracy is not satisfactory. This is the case for example when the buffer sizes are too small in comparison with the mean times to repair or in the case of close-loop production lines. We present in this paper a new decomposition method that is a better approximation of the repair time distribution.

The new method also characterizes the repair time distribution of the equivalent machines by a GE distribution. However, it differs from the GE-method in the way the parameters of the GE distribution are obtained. In this paper, we compare the accuracy of the new method with that of Gershwin's original method, referred to as the exponential method (E-method), and that of the GE-method proposed in [8]. Since the exponential distribution is a special case of a GE distribution, all three methods can be viewed as different ways of approximating the repair time distributions of the equivalent machines by a GE distribution. Except for this feature, the three methods use the same core decomposition procedure. The three method can readily be incorporated in the decomposition algorithm proposed in [10] for closed-loop production lines.

The paper is organized as follows. The continuous flow model is presented in Section 2. The general decomposition method is described in Section 3. The general computational algorithm is given in Section 4. The different methods namely the FM-method, the TM-method and the BM-method, are discussed in section 5. Finally, numerical examples pertaining to both open and closed-loop production lines are reported in Section 6.

2. The Continuous Flow Model

We consider the continuous flow model of a production line consisting of a series of K machines (M_1, M_2, ..., M_K) separated by K-1 intermediate buffers (B_1, B_2, ..., B_{K-1}). Parts flow from outside the system to machine M_1, then to buffer B_1, then to machine M_2, and so forth until they reach machine M_K, after which they leave the system. We assume that there are always parts available at the input of the system and space available at the output of the system. The intermediate buffers are each of finite capacity (C_1, C_2, ..., C_{K-1}). A four machine production line is shown in Figure 1.

The quantity of material in each buffer B_i at any time is a real number $h_i(t)$, where $0 \leq h_i(t) \leq C_i$. Each machine can be in two states: operational (not in a failure condition) or down (under repair). When operational, it can be either working or idle. A machine is idle if it is starved or blocked. Machine M_i is starved at time t if one of the upstream machines is down and all buffers between this machine and machine M_i are empty. Machine M_i is blocked if one of the downstream machines is down and all the buffers in between this machine and machine M_i are full. A machine that is operational and neither starved nor blocked is working. When machine M_i is working, it transfers material from its upstream buffer, B_{i-1}, to its downstream buffer, B_i, at a constant rate U. That is a quantity of material Udt is transferred in time dt. Note that U=1/T, where T is the processing time of all the machines in the original asynchronous model.

A machine may fail only while it is working. The time to failure and the time to repair of machine M_i are exponentially distributed with rates λ_i and μ_i, respectively. Throughout the rest of the paper we assume that T = 1 and that, as a result U = 1. This is without loss of generality since the time unit can always be chosen so that this condition is satisfied. Let L denote the continuous flow model of the production line.

Let us define the following performance parameters of the continuous model:

e_i: Isolated efficiency of machine M_i.

$$e_i = \frac{\mu_i}{\lambda_i + \mu_i} = \frac{1}{1 + \dfrac{\lambda_i}{\mu_i}}$$

E_i: Efficiency of machine M_i; proportion of time machine M_i is working in line L.

ps_i: Probability of machine M_i being starved in line L.

pb_i: Probability of machine M_i being blocked in line L.

Since $U = 1$, E_i can equivalently be interpreted as the production rate of machine M_i in line L. The following relationships have been established in [6] for the continuous flow model of tandem production lines.

$$E_1 = E_2 = ... = E_{K-1} \qquad (1)$$
$$E_i = e_i (1 - ps_i - pb_i) \qquad \text{for any } i = 1, ..., K-1 \qquad (2)$$

The first relation is related to conservation of flow. The second equation relates the efficiency of machine M_i in line L, E_i, to its efficiency when considered in isolation e_i. It is an exact relationship in the case of the continuous flow model because a machine cannot be simultaneously starved and blocked.

3. Decomposition Method

In this section we present the general decomposition method for the analysis of the continuous flow model. The principle of this method is to decompose the K-machine line into a set of K-1 two-machine lines L(i), for i = 1, ..., K-1 (see Figure 1). Each line L(i) is a continuous flow model consisting of an upstream machine $M_u(i)$, a downstream machine, $M_d(i)$, and an intermediate buffer, B(i). There is an infinite supply of material in front of machine $M_u(i)$, i.e, $M_u(i)$ is never starved, while there is an infinite amount of storage available at the output of machine $M_d(i)$, i.e, $M_d(i)$ is never blocked. Both machines have the same processing rate as the machines of line L, that is U = 1. The capacity of buffer B(i) is the same as that of buffer B_i, that is C_i. We assume that the failure times of machine $M_u(i)$ and $M_d(i)$ have exponential distribution with parameters $\lambda_u(i)$ and $\lambda_d(i)$, respectively, while their repair times have generalized exponential distributions with parameters $(\mu_u(i), q_u(i))$ and $(\mu_d(i), q_d(i))$ respectively. Recall that a random variable X has a generalized exponential (GE) distribution [14] if it has the following cumulative distribution function:

$$F_X(x) = P [X \le x] = (1-q) + q(1-e^{-\mu x}),\ x \ge 0, \text{ where } 0 < q \le 1 \text{ and } \mu > 0$$

This means that, with a probability q, the service time is exponentially distributed with rate μ, and with a probability 1-q, the service time is zero.

Each system L(i) must be defined in such a way that the behavior of the material flow in buffer B(i) closely matches that of the material flow in buffer B_i of line L. Machine $M_u(i)$ represents (in an aggregate way) the part of the line upstream of buffer B_i, while machine $M_d(i)$ represents (in an aggregate way) the part of the line downstream of buffer B_i (upstream and downstream refer to the direction of the flow of material). In other words, machine $M_u(i)$ models how material is transferred into buffer B_i, while machine $M_d(i)$ models how material is transferred out of buffer B_i. In order to determine the unknown parameters $\lambda_u(i)$, $\mu_u(i)$, $q_u(i)$, $\lambda_d(i)$, $\mu_d(i)$, $q_d(i)$, for each two-machine line L(i), i=1, ..., K-1, we need a set of 6(K-1) independent equations. In the sequel of this section we derive some basic equations that are common to the three different decomposition method considered in this paper. Additional equations specific to each particular method will be derived in section 5.

Let us define the following quantities:

$$e_u(i) = \frac{1}{1+\frac{q_u(i)\lambda_u(i)}{\mu_u(i)}} = \frac{\mu_u(i)}{\mu_u(i)+q_u(i)\lambda_u(i)} \qquad i = 1, ..., K\text{-}1 \qquad (3)$$

$$e_d(i) = \frac{1}{1+\frac{q_d(i)\lambda_d(i)}{\mu_d(i)}} = \frac{\mu_d(i)}{\mu_d(i)+q_d(i)\lambda_d(i)} \qquad i = 1, ..., K\text{-}1 \qquad (4)$$

$e_u(i)$ and $e_d(i)$ represent the efficiencies of machines $M_u(i)$ and $M_d(i)$ in isolation.
For each line L(i), we define the following performance parameters:

E(i): Efficiency of line L(i). This is the proportion of time machine $M_d(i)$ is working.
$p_s(i)$: Probability of machine $M_d(i)$ being starved in line L(i).
$p_b(i)$: Probability of machine $M_u(i)$ being blocked in line L(i).

Again since U = 1, E(i) can equivalently be interpreted as the production rate of line L(i). These performance parameters are functions of the parameters of the upstream and downstream machines of L(i).We have the following relationships [7]:

$$E(i) = e_u(i)\,(1 - p_b(i)) \qquad i = 1, ..., K\text{-}1 \qquad (5)$$
$$E(i) = e_d(i)\,(1 - p_s(i)) \qquad i = 1, ..., K\text{-}1 \qquad (6)$$

From equation (1) and (2) and the above conditions the following set of equations can be derived (see [6,7] for details):

$$E(1) = E(2) = ... = E(K\text{-}1) \qquad (7)$$
$$E(i\text{-}1) = e_i\,(1 - p_s(i\text{-}1) - p_b(i)) \qquad i = 2, ..., K\text{-}1 \qquad (8)$$

Now using (5), (6) and (7), after some manipulations, (8) can be written:

$$\frac{1}{e_d(i-1)} + \frac{1}{e_u(i)} = \frac{1}{E(i-1)} + \frac{1}{e_i} \qquad i = 2, ..., K\text{-}1 \qquad (9)$$

Let us now consider the failure/repair mechanism of the equivalent machines. A failure of machine $M_u(i)$ represents either a failure or a starvation of machine M_i. A starvation of machine M_i is a consequence of either a failure or a starvation of machine M_{i-1}. Now, a failure or a starvation of machine M_{i-1} is represented by a failure of machine $M_u(i\text{-}1)$. Therefore, a failure of machine $M_u(i)$ results from either a failure of machine M_i or a failure of machine $M_u(i\text{-}1)$. Let $\alpha_u(i)$ be the proportion of time the cause of the failure of machine $M_u(i)$ is a failure of machine $M_u(i\text{-}1)$. As a result, the repair time of machine $M_u(i)$ is either the repair time of machine M_i, or the residual repair time of machine $M_u(i\text{-}1)$ at the instant of starvation of machine M_i. Thus, the repair time distribution of machine $M_u(i)$ is a probabilistic mixture of the residual repair time distribution of machine $M_u(i\text{-}1)$ (with probability $\alpha_u(i)$) and of the repair time distribution of machine M_i (with probability $1\text{-}\alpha_u(i)$). Let $m_u^{(1)}(i)$ be the mean time to repair of machine $M_u(i)$ and let $m_u^r(i-1)$ the average residual repair time of machine $M_u(i\text{-}1)$ at the instant of starvation. As a result, the quantity $m_u^{(1)}(i)$ is given by:

$$m_u^{(1)}(i) = \alpha_u(i)m_u^r(i-1) + (1-\alpha_u(i))m_i^{(1)} \qquad i = 2, ..., K-1 \qquad (10)$$

where $m_i^{(1)}$ is the mean time to repair of machine M_i with $m_i^{(1)} = \frac{1}{\mu_i}$. Let us note that the residual service time of a GE distribution with parameters (μ, q), after any positive time, is simply an exponential distribution with rate μ. Thus we have:

$$m_u^r(i-1) = \frac{1}{\mu_u(i-1)} \qquad i = 2, ..., K-1 \qquad (11)$$

As a result, (10) can be written as:

$$m_u^{(1)}(i) = \frac{\alpha_u(i)}{\mu_u(i-1)} + \frac{1-\alpha_u(i)}{\mu_i} \qquad i = 1, ..., K-1 \qquad (12)$$

The repair time distribution of machine $M_u(i)$ is an hyper-exponential distribution with two stage. That is with a probability $\alpha_u(i)$ the repair time of $M_u(i)$ is an exponential time with rate $\mu_u(i-1)$, and with probability 1- $\alpha_u(i)$, it is an exponential time with rate μ_i. This is illustrated in Figure 2. All three methods approximate the hyper-exponential distribution by a GE-distribution with parameters $\mu_u(i)$ and $q_u(i)$. Two equations are needed to determine the unknown parameters $\mu_u(i)$ and $q_u(i)$. One equation is common to all three methods. Namely it prescribes that the GE-distribution has the same mean as the original hyper-exponential distribution. The difference between the three methods pertains to the second equation. This will discussed in section 5.

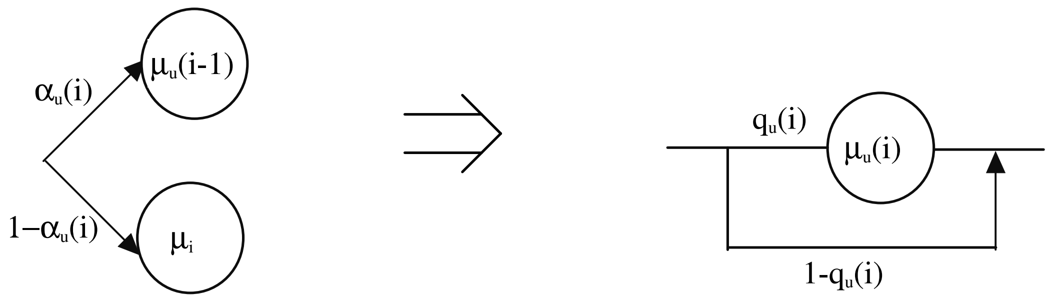

Figure 2. Illustration of the repair time distribution of machine $M_u(i)$: the original hyper-exponential distribution and its approximation by an GE-distribution

The mean time to repair of the GE repair time distribution is given by the following relationship:

$$\frac{q_u(i)}{\mu_u(i)} = m_u^{(1)}(i) \qquad (13)$$

Using (13), (12) can be written as:

$$\frac{q_u(i)}{\mu_u(i)} = \frac{\alpha_u(i)}{\mu_u(i-1)} + \frac{1-\alpha_u(i)}{\mu_i} \qquad i = 2, ..., K\text{-}1 \qquad (14)$$

Let us now calculate the quantity $\alpha_u(i)$. $\alpha_u(i)$ is the ratio of the average number of failures (per unit of time) of machine $M_u(i)$ which results from a failure of machine $M_u(i\text{-}1)$ to the average number of failures (per unit of time) of machine $M_u(i)$. Let $D_u(i)$ be the probability of machine $M_u(i)$ being in a failure condition. Then, the average number of failures (per unit of time) occurring on machine $M_u(i)$, which is equal to that of repairs, is equal to $D_u(i)/m_u^{(1)}(i)$. On the other hand, the probability of machine $M_u(i)$ being in a failure condition which is a consequence of a failure of machine $M_u(i\text{-}1)$, is equal to $p_s(i\text{-}1)$. Therefore, the average number of failures (per unit of time) of $M_u(i)$ which results from a failure of machine $M_u(i\text{-}1)$, which is equal to that of repairs, is equal to $p_s(i\text{-}1)/m_u^r(i-1)$. As a result the quantity $\alpha_u(i)$ is given by:

$$\alpha_u(i) = \frac{p_s(i-1)m_u^{(1)}(i)}{D_u(i)m_u^r(i-1)} \qquad i = 2, ..., K\text{-}1 \qquad (15)$$

The probability of machine $M_u(i)$ being in failure condition, $D_u(i)$, can be expressed as (see Property 1 in the Appendix of [7]):

$$D_u(i) = (\frac{1}{e_u(i)} - 1)E(i) \qquad i = 1, ..., K\text{-}1 \qquad (16)$$

Let $A_u(i)$ be defined as:

$$A_u(i) = \frac{p_s(i-1)}{D_u(i)} = \frac{p_s(i-1)}{(\frac{1}{e_u(i)} - 1)E(i)} \qquad i = 2, ..., K\text{-}1 \qquad (17)$$

Using (15), (17) and (11), after some manipulations, (14) leads to:

$$\frac{\mu_u(i)}{q_u(i)} = A_u(i)\mu_u(i-1) + (1 - A_u(i))\mu_i \qquad i = 2, ..., K\text{-}1 \qquad (18)$$

A similar analysis of the failure-repair mechanism of machine $M_d(i)$ yields the following equations:

$$\frac{\mu_d(i)}{q_d(i)} = A_d(i)\mu_d(i+1) + (1 - A_d(i))\mu_{i+1} \qquad i = 1, ..., K\text{-}2 \qquad (19)$$

where $A_d(i) = \frac{p_b(i+1)}{D_d(i)} = \frac{p_b(i+1)}{(\frac{1}{e_d(i)} - 1)E(i)} \qquad i = 1, ..., K\text{-}2 \qquad (20)$

Finally, there are boundary conditions:

$\lambda_u(1) = \lambda_1$; $\mu_u(1) = \mu_1$; $q_u(1) = 1$; (21)

$\lambda_d(K\text{-}1) = \lambda_K$; $\mu_d(K\text{-}1) = \mu_K$; $q_d(K\text{-}1) = 1$; (22)

4. Computational Algorithm

In this section, we propose a general algorithm to determine the unknown parameters of the decomposition method, which is the generalization of the DDX algorithm [5,6]. First of all, it is important to recognize that a two-machine production line model with failure and repair time distributions characterized by exponential and GE distributions, respectively, can be analyzed exactly. Indeed, it is shown in [4] that such a model can be transformed into an equivalent production line model with both failure and repair time distributions characterized by exponential distributions for which, as recalled in the introduction, an exact solution can easily be performed [9,12]. The transformation in [4] relevant for this paper is as follows: a machine with exponential failure time distribution with parameter λ and GE repair time distribution with parameters (μ, q) is equivalent to a machine with exponential failure time distribution with parameter $q\lambda$ and exponential repair time distribution with parameter μ. This implies that for given values of the parameters $\lambda_u(i)$, $\mu_u(i)$, $q_u(i)$, $\lambda_d(i)$, $\mu_d(i)$ and $q_d(i)$, of line L(i), the performance parameters of interest, E(i), $p_s(i)$ and $p_b(i)$ can easily be derived. Namely, they can be obtained by solving an equivalent two-machine line with exponential failure and repair time distributions whose parameters $\lambda_u^e(i)$, $\mu_u^e(i)$, $\lambda_d^e(i)$, $\mu_d^e(i)$, are given by:

$$\lambda_u^e(i) = q_u(i)\lambda_u(i) \qquad \lambda_d^e(i) = q_d(i)\lambda_d(i)$$
$$\mu_u^e(i) = \mu_u(i) \qquad \mu_d^e(i) = \mu_d(i)$$

In order to obtain the final computational algorithm, let us transform the original set of equations. After some manipulations, the 2(K-2) equations given by (7) and (9) can be equivalently transformed into the following equations:

$$\frac{1}{e_u(i)} = \frac{1}{E(i-1)} + \frac{1}{e_i} - \frac{1}{e_d(i-1)} \qquad i = 2, ..., K\text{-}1 \qquad (23)$$

$$\frac{1}{e_d(i)} = \frac{1}{E(i+1)} + \frac{1}{e_{i+1}} - \frac{1}{e_u(i+1)} \qquad i = 1, ..., K\text{-}2 \qquad (24)$$

Using (7), (17) and (20) can be equivalently transformed into the following equations:

$$A_u(i) = \frac{p_s(i-1)}{(\frac{1}{e_u(i)} - 1)E(i-1)} \qquad i = 2, ..., K\text{-}1 \qquad (25)$$

$$A_d(i) = \frac{p_b(i+1)}{(\frac{1}{e_d(i)} - 1)E(i+1)} \qquad i = 1, ..., K\text{-}2 \qquad (26)$$

Algorithm

Step 1. Set:

$\lambda_u(1) = \lambda_1$; $\mu_u(1) = \mu_1$; $q_u(1) = 1$;

$\lambda_d(K-1) = \lambda_K$; $\mu_d(K-1) = \mu_K$; $q_d(K-1) = 1$;

Initialize

$\lambda_d(i) = \lambda_{i+1}$; $\mu_d(i) = \mu_{i+1}$; $q_d(i) = 1$; $\quad$ i = 1, ..., K-2

Step 2. For i=2, 3,..., K-1, calculate:

$$\frac{1}{e_u(i)} = \frac{1}{E(i-1)} + \frac{1}{e_i} - \frac{1}{e_d(i-1)}$$

$$A_u(i) = \frac{p_s(i-1)}{(\frac{1}{e_u(i)} - 1)E(i-1)}$$

Calculate the GE parameters ($\mu_u(i)$, $q_u(i)$) of the repair time distribution of $M_u(i)$

$$\lambda_u(i) = (\frac{1}{e_u(i)} - 1)\frac{\mu_u(i)}{q_u(i)}$$

Step 3. For i = K-2, K-3,..., 1, calculate:

$$\frac{1}{e_d(i)} = \frac{1}{E(i+1)} + \frac{1}{e_{i+1}} - \frac{1}{e_u(i+1)}$$

$$A_d(i) = \frac{p_b(i+1)}{(\frac{1}{e_d(i)} - 1)E(i+1)}$$

Calculate the GE parameters ($\mu_d(i)$, $q_d(i)$) of the repair time distribution of $M_d(i)$

$$\lambda_d(i) = (\frac{1}{e_d(i)} - 1)\frac{\mu_d(i)}{q_d(i)}$$

Step 4. Go to Step 2 until convergence of the unknown parameters

Different approaches to determine the parameters $\mu_u(i)$, $q_u(i)$, $\mu_d(i)$ and $q_d(i)$, are presented in section 5. The difference between the three methods is the approximation method of the repair time distribution of the upstream and the downstream machines of line L(i) from a hyper-exponential distribution to a GE distribution.

5. Approximation of the repair time distribution

We now discuss the three different methods to determine the parameters of the GE-distribution of the repair times of the equivalent machines. Recall that from the analysis of the decomposition method we ended up with a first characterization of the repair time distributions as hyper-exponential distributions (see Figure 2) Each method consists in the specification of an approach to construct a GE-distribution that approximately represents the original hyper-exponential distribution. Consider successively the following three different methods: a first-moment approximation, a two-moment approximation, and a behavior-matching approximation. All three methods ensure that the mean repair time of the constructed GE-distribution is the same as that of the original hyper-exponential distribution. Thus, equations (18) and (19) are used in all three methods.

5.1 First-moment Approximation

In the first approach used in [6] the repair time distributions are characterized by exponential distributions. This implies that the GE-distributions are such that $q_u(i) = q_d(i) = 1$. Equations (18) and (19) can then simply be used to determine the second parameters of the GE-distributions, namely $\mu_u(i)$ and $\mu_d(i)$. Thus we end up with the following set of equation for machine $M_u(i)$:

$$q_u(i)=1 \qquad i = 1, ..., K\text{-}1 \qquad (27)$$

$$\mu_u(i) = A_u(i)\mu_u(i-1) + (1 - A_u(i))\mu_i \qquad i = 2, ..., K\text{-}1 \qquad (28)$$

and for machine $M_d(i)$:

$$q_d(i)=1 \qquad i = 1, ..., K\text{-}1 \qquad (29)$$

$$\mu_d(i) = A_d(i)\mu_d(i+1) + (1 - A_d(i))\mu_{i+1} \qquad i = 1, ..., K\text{-}2 \qquad (30)$$

5.2 Two-moment Approximation

A GE distribution can be used as a two moment approximation of any distribution, provided that the coefficient of variation of that distribution is greater or equal to one. Let $m^{(1)}$ and $m^{(2)}$ be the first two moments of the distribution to be approximated. Then, the parameters of the "equivalent" GE distribution are obtained as the solution of [4]:

$$\frac{q}{\mu} = m^{(1)}; \qquad \frac{2q}{\mu^2} = m^{(2)}$$

Now, since the hyper-exponential distribution has a coefficient of variation greater or equal to one, it can always be approximated by a GE-distribution having the same first two-moments, that is the same mean and variance.

The first moment equation is given by equation (12). But we have shown in part 3 that (12) can equivalently be written:

$$\frac{\mu_u(i)}{q_u(i)} = A_u(i)\mu_u(i-1) + (1 - A_u(i))\mu_i \qquad i = 2, ..., K\text{-}1 \qquad (18)$$

The second moment equation can be expressed as:

$$\frac{2q_u(i)}{\mu_u(i)^2} = \frac{2\alpha_u(i)}{\mu_u(i-1)^2} + \frac{2(1-\alpha_u(i))}{\mu_i{}^2} \tag{31}$$

Using (18) and (17) , after some manipulations, (31) become:

$$\frac{1}{\mu_u(i)} = \frac{A_u(i)}{\mu_u(i-1)} + \frac{1-A_u(i)}{\mu_i} \qquad i = 2, ..., K\text{-}1 \tag{32}$$

(18) can be equivalently be written as :

$$\frac{1}{q_u(i)} = \frac{A_u(i)\mu_u(i-1) + (1-A_u(i))\mu_i}{\mu_u(i)} \qquad i = 2, ..., K\text{-}1 \tag{33}$$

A similar analysis of the failure-repair mechanism of machine $M_d(i)$ yields the following equations:

$$\frac{1}{\mu_d(i)} = \frac{A_d(i)}{\mu_d(i+1)} + \frac{1-A_d(i)}{\mu_{i+1}} \qquad i = 1, ..., K\text{-}2 \tag{34}$$

and

$$\frac{1}{q_d(i)} = \frac{A_d(i)\mu_d(i+1) + (1-A_d(i))\mu_{i+1}}{\mu_d(i)} \qquad i = 1, ..., K\text{-}2 \tag{35}$$

Thus equations (32) and (33) (equations (34) and (35), respectively) can be used to determine the parameters of the GE-distribution of machine $M_u(i)$ (machine $M_d(i)$, respectively) that has the same first two-moments as those of the original hyper-exponential distribution.

5.3 Behavior-matching Approximation

The first two methods presented in Sections 5.1 and 5.2 used a moment-matching approach. The third method uses a different approach that we refer to as a behavior-matching approach. In the general decomposition method discussed in Section 3, the distributions of failure times of the equivalent machines are characterized by exponential distributions, while the distributions of repair times are characterized by GE distributions.

Let us introduce two new two-machine lines: $L^u(i)$ is a two-machine line identical to line L(i), except that the repair time distribution of machine $M_u(i)$ is characterized by an hyper-exponential distribution with parameters $\alpha_u(i)$, $\mu_u(i\text{-}1)$ and μ_i (see Figure 2). Similarly, $L^d(i)$ is a two-machine line identical to line L(i), except that the repair time distribution of machine $M_d(i)$ is characterized by an hyper-exponential distribution with parameters $\alpha_d(i)$, $\mu_d(i+1)$ and μ_{i+1}. The idea is to determine the parameters of the GE-distribution of the repair time of machine $M_u(i)$ (machine $M_d(i)$ respectively) in such a way that the production rate of line L(i) is equal to that of line $L^u(i)$ (line $L^d(i)$ respectively), which is equivalent to prescribing that the two lines have the same efficiency (since U = 1 in both lines). Let $E^u(i)$ denote the efficiency of line

$L^u(i)$, and $E^d(i)$ the efficiency of line $L^d(i)$. The two equations then used to determine the parameters of the GE-distribution of machine $M_u(i)$ are equation (18) and the following equation:

$$E(i) = E^u(i) \tag{36}$$

Similarly, the two equations used to determine the parameters of the GE-distribution of machine $M_d(i)$ are equation (19) and the following equation:

$$E(i) = E^d(i) \tag{37}$$

For given parameters of line $L^u(i)$ ($L^d(i)$, respectively), it is possible to analyze its behavior and calculate its efficiency $E^u(i)$ ($E^d(i)$, respectively). Such an analysis is reported in [15]. The parameters of the GE repair time distribution of machine $M_u(i)$, $\mu_u(i)$ and $q_u(i)$, can then be calculated by solving equations (18) and (36). The solution is given below (see [15] for details):

$$q_u(i) = \frac{-W_u + \sqrt{\Delta_u}}{2\left(\frac{1 - e_u(i)V_u}{1 - e_u(i)}\right)\lambda_u(i)} \tag{38}$$

$$V_u = \frac{\log\left(\frac{e_d(i)^2(1 - e_u(i))(E^u(i) - e_u(i))}{e_u(i)^2(1 - e_d(i))(E^u(i) - e_d(i))}\right)}{C(i)q_d(i)\lambda_d(i)\left(\frac{e_u(i)}{1 - e_u(i)} - \frac{e_d(i)}{1 - e_d(i)}\right)} \tag{39}$$

$$W_u = q_d(i)\lambda_d(i)\left(1 - \frac{e_u(i)V_u}{1 - e_u(i)} + \frac{e_d(i)(1 - V_u)}{1 - e_d(i)}\right) \tag{40}$$

$$\Delta_u = W_u{}^2 + 4\left(\frac{1 - e_u(i)V_u}{1 - e_u(i)}\right)V_u \frac{(q_d(i)\lambda_d(i))^2}{I_d(i)} \tag{41}$$

$$\mu_u(i) = q_u(i)\big(A_u(i)\mu_u(i-1) + (1 - A_u(i))\mu_i\big) \qquad i = 2, ..., K\text{-}1 \tag{42}$$

In the special case where eu(i) = ed(i) = e(i), equations (38) must be replaced by:

$$q_u(i) = \frac{q_d(i)\lambda_d(i)(1 - e(i))\big((2 - e(i))E^u(i) - e(i)\big)}{\lambda_u(i)\big[q_d(i)\lambda_d(i)C(i)\big(e(i) - E^u(i)\big) + (1 - e(i))\big(e(i) - (2 - e(i))E^u(i)\big)\big]} \tag{43}$$

A similar analysis yields the following results for the determination of the parameters of the GE repair time distribution of machine $M_d(i)$, $\mu_d(i)$ and $q_d(i)$:

$$q_d(i) = \frac{-W_d + \sqrt{\Delta_d}}{2\left(\frac{1-e_d(i)V_d}{1-e_d(i)}\right)\lambda_d(i)} \quad (44)$$

$$V_d = \frac{\log\left(\frac{e_u(i)^2(1-e_d(i))(E^d(i)-e_d(i))}{e_d(i)^2(1-e_u(i))(E^d(i)-e_u(i))}\right)}{C(i)q_u(i)\lambda_u(i)\left(\frac{e_d(i)}{1-e_d(i)}-\frac{e_u(i)}{1-e_u(i)}\right)} \quad (45)$$

$$W_d = q_u(i)\lambda_u(i)\left(1-\frac{e_d(i)V_d}{1-e_d(i)}+\frac{e_u(i)(1-V_d)}{1-e_u(i)}\right) \quad (46)$$

$$\Delta_d = W_d{}^2 + 4\left(\frac{1-e_d(i)V_d}{1-e_d(i)}\right)V_d\frac{(q_u(i)\lambda_u(i))^2}{I_u(i)} \quad (47)$$

$$\mu_d(i) = q_d(i)\big(A_d(i)\mu_d(i+1)+(1-A_d(i))\mu_{i+1}\big) \qquad i = 1, ..., K\text{-}2 \quad (48)$$

Again, in the special case where eu(i) = ed(i) = e(i), equations (44) must be replaced by:

$$q_d(i) = \frac{q_u(i)\lambda_u(i)(1-e(i))\big((2-e(i))E^d(i)-e(i)\big)}{\lambda_d(i)\big[q_u(i)\lambda_u(i)C(i)\big(e(i)-E^d(i)\big)+(1-e(i))\big(e(i)-(2-e(i))E^d(i)\big)\big]} \quad (49)$$

6. Numerical Experiments

In this section, we compare the results obtained using the three methods presented in section 5.1, 5.2, 5.3. Those methods will be referred to as the First-moment (FM) method, the two-moment (TM) method, and the behavior-matching (BM) method. We compare these results with simulation results of the continuous flow model of the original production lines. We ran long enough simulations so that the outcome of these simulations can be used as a reference to compare the three decomposition methods. For each example, we provide the average buffer levels as well as the production rate of the line. We also provide the relative error of each decomposition method with respect to the simulation results.

The first example (Example O1) pertains to a line consisting of 10 machines. The basic configuration (O1) is an open production line. The configuration (C1) is a closed-loop production line. The machine parameters of the closed-loop production line are the same as those of the configuration (O1). For the configuration (O1) it appears that the GE-method and the new method provide accurate results in the evaluation of the production rate of the original line. The poor results of the E-method come from the fact that the repair time of the original machines are significantly different. Thus the first moment approximation of the repair time distribution in the case of the decomposition method is not accurate. In the case of the closed loop production line (C1), it appears that the new decomposition method provides more accurate results.

The configurations (O2), (O3), ..., and (O7) are symmetrical case of (open) production lines with five machines. In the case of the configuration (O2), (O4), (O6) the buffer sizes are equal to 100. For the other configurations of open production line the buffer size are equal to 20. For all examples of open production lines, the A-configuration is such that all machines of the line have the same efficiency in isolation, i.e.: $e_1 = ... = e_K$; however, the failure and repair rates differ by an order of magnitude. In the configuration (O2-A) and (O3-A) all machines of the line are the same. The two other pairs of configuration, (O4-A) and (O5-A), (O6-A) and (O7-A) are obtained by changing the order of magnitude of the repair times and the failure times of machines 2 and 4. The configuration (OX-B) is derived from the configuration (OX-A) by changing the order of magnitude of the repair time of machines 2 and 4. The data of these examples are given from table 6 to table 10 and the results are provides from table 11 to table 16. The results obtained using the two decomposition methods are identical for the configuration (O2-A) and (O3-A). This is not surprising since it is easy to show that the three decomposition methods are equivalent when the exponential distributions of the repair times in the original lines are all the same, i.e.: $\mu_1 = ... = \mu_K$. For all other configurations, it appears that the GE and the new decomposition methods provide more accurate results than those obtained with the E decomposition method. In the case of configurations (O6-A), (O6-B), and (O7-B), the average buffer levels evaluated with the E-method do not respect the symmetry of the line. This is discussed in [8].

The configurations (C2) and (C3) are symmetrical case of closed-loop production lines with 10 machines. The machines parameters of the configurations (C2) and (C3) are respectively similar t to those of configurations (O4) and (O5). The data of these examples are given in tables 17 and 18 and the results are provides from table 19 to table 22. In the case of closed-loop production line the new method appear to be the most robust one. If the buffer sizes are equal to 100, and the number of pallets in the system is equal to 500 the GE-method provides very accurate results. It corresponds to a well designed system. The results of the GE method are not so good if the number of pallets is equal to 200 or the buffer size is equal to 20. Generally the E-method gives poor results.

Let us briefly discuss the convergence and computational complexity of this algorithm. Generally speaking, the behavior of the GE algorithm is very similar to that of the original FM-DDX algorithm. We tested many examples with quite different parameters and on all these examples, the TM-DDX algorithm and the BM-DDX algorithm always converged. The number of iterations to achieve convergence is in general very low and always close to that required by the FM-DDX algorithm. Since the analysis of two-machine model with TM-DDX distribution of repair times has the same complexity as in the FM-DDX algorithm (as explained in Section 4), we conclude that at least our algorithm has the same computational complexity as the FM-DDX algorithm. The case of the BM-DDX algorithm is different because the analysis of a two-machine model with an hyper-exponential distribution is more slightly difficult.

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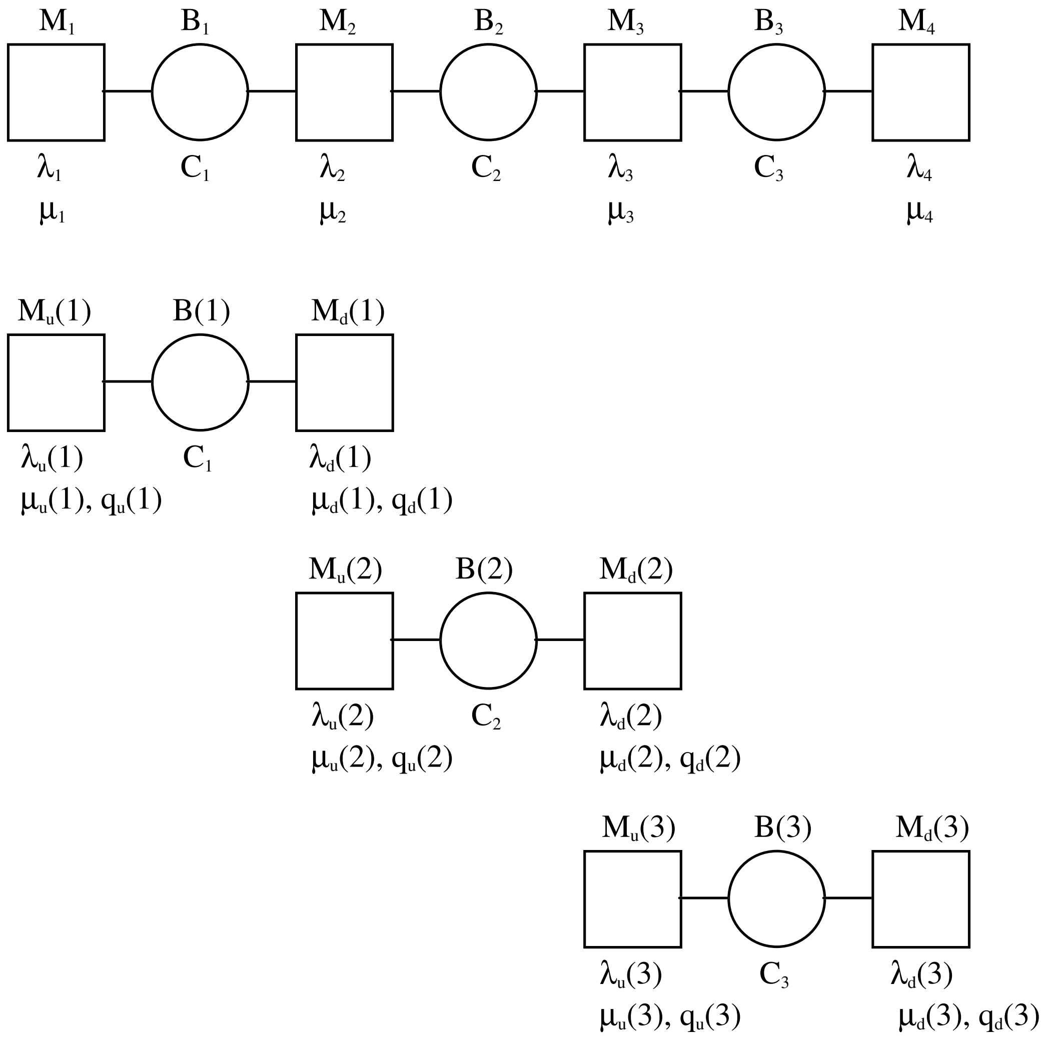

Fig. 1 *A four-machine production line*
and its decomposition into three two-machine production lines

		1	2	3	4	5	6	7	8	9	10
O1	$1/\lambda i$	50	400	150	200	400	100	500	600	100	800
	$1/\mu i$	5	60	10	15	40	5	40	60	5	60
	ei	0,909	0,870	0,938	0,930	0,909	0,95	0,926	0,909	0,952	0,930
	Ui	1	1	1	1	1	1	1	1	1	1
	Ci	50	60	20	30	30	50	70	50	60	

Table 1: Data for Example 01

		1	2	3	4	5	6	7	8	9	10
C1	$1/\lambda i$	50	400	150	200	400	100	500	600	100	800
	$1/\mu i$	5	60	10	15	40	5	40	60	5	60
a) 96 pallets	ei	0,909	0,870	0,938	0,930	0,909	0,95	0,926	0,909	0,952	0,930
b) 240 pallets	Ui	1	1	1	1	1	1	1	1	1	1
	Ci	50	60	20	30	30	50	70	50	60	60

Table 2: Data for Example C1

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	X
O1	simulation	34,2268	36,9555	10,9399	13,0374	11,2069	17,5750	22,3478	12,5643	11,6938	0,7919
	+/-	0,0077	0,0013	0,0034	0,0061	0,0060	0,0113	0,0171	0,0093	0,0109	0,0001
	FM-method	31,2972	31,2859	9,6343	11,8810	11,2543	13,8858	18,8962	12,7172	9,8993	0,8227
	error(%)	**-8,56**	**-15,34**	**-11,93**	**-8,87**	**0,42**	**-20,99**	**-15,44**	**1,22**	**-15,35**	**3,89**
	TM-method	33,1647	38,3790	11,5549	12,7199	10,0923	15,2029	19,7857	12,3587	10,4235	0,7854
	error(%)	**-3,10**	**3,85**	**5,62**	**-2,44**	**-9,95**	**-13,50**	**-11,46**	**-1,64**	**-10,86**	**-0,82**
	BM-method	33,4888	37,2414	10,5435	11,9242	10,5504	15,0319	20,0329	12,8681	10,3185	0,7991
	error(%)	**-2,16**	**0,77**	**-3,62**	**-8,54**	**-5,86**	**-14,47**	**-10,36**	**2,42**	**-11,76**	**0,91**

Table 3: Comparison of results obtained using the three decomposition methods (example 01)

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C1 a)	simulation	13,1901	14,4503	6,6067	7,0018	5,3354	8,5531	10,9345	5,9948	7,6796	16,2537	0,7320
	+/-	0,0072	0,0073	0,0027	0,0040	0,0028	0,0058	0,0087	0,0033	0,0065	0,0061	0,0001
	FM-method	13,1704	11,4431	5,5029	6,6082	6,2768	9,4301	11,6263	5,2954	8,6738	17,9729	0,7686
	error(%)	**-0,15**	**-20,81**	**-16,71**	**-5,62**	**17,64**	**10,25**	**6,33**	**-11,67**	**12,95**	**10,58**	**5,00**
	TM-method	12,2252	13,5811	5,7349	6,4680	5,0547	8,0538	10,5724	7,2312	10,2348	16,8440	0,6914
	error(%)	**-7,32**	**-6,02**	**-13,20**	**-7,62**	**-5,26**	**-5,84**	**-3,31**	**20,62**	**33,27**	**3,63**	**-5,55**
	BM-method	12,1344	12,7133	5,5192	6,2253	5,2340	8,2956	11,0586	7,2064	10,1375	17,4557	0,7178
	error(%)	**-8,00**	**-12,02**	**-16,46**	**-11,09**	**-1,90**	**-3,01**	**1,13**	**20,21**	**32,01**	**7,40**	**-1,94**

Table 4: Comparison of results obtained using the three decomposition methods (example C1 with 96 pallets)

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C1 b)	simulation	27,5667	31,9311	10,2171	12,1289	10,5512	17,8293	27,6898	23,6840	35,7377	42,6642	0,7714
	+/-	0,0092	0,0125	0,0034	0,0056	0,0054	0,0102	0,0146	0,0104	0,0130	0,0112	0,0001
	FM-method	26,0162	26,6642	8,9713	10,8520	10,0630	14,1724	27,2226	25,6096	41,8284	48,6003	0,8211
	error(%)	**-5,62**	**-16,49**	**-12,19**	**-10,53**	**-4,63**	**-20,51**	**-1,69**	**8,13**	**17,04**	**13,91**	**6,44**
	TM-method	26,8007	31,9375	10,5176	12,3221	10,5088	18,1495	29,9292	24,8232	34,1407	40,8707	0,7662
	error(%)	**-2,78**	**0,02**	**2,94**	**1,59**	**-0,40**	**1,80**	**8,09**	**4,81**	**-4,47**	**-4,20**	**-0,67**
	BM-method	26,2564	31,0839	9,7893	11,5258	10,6834	17,3612	29,5286	25,4065	35,3276	43,0373	0,7817
	error(%)	**-4,75**	**-2,65**	**-4,19**	**-4,97**	**1,25**	**-2,63**	**6,64**	**7,27**	**-1,15**	**0,87**	**1,34**

Table 5: Comparison of results obtained using the two decomposition methods (example C1 with 240 pallets)

		1	2	3	4	5
	$1/\lambda i$	1000	1000	1000	1000	1000
O2-A	$1/\mu i$	100	100	100	100	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O2-B	$1/\mu i$	100	10	100	10	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	100	100	100	100	

Table 6: Data for Example 02

		1	2	3	4	5
	$1/\lambda i$	1000	1000	1000	1000	1000
O3-A	$1/\mu i$	100	100	100	100	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O3-B	$1/\mu i$	100	10	100	10	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	20	20	20	20	

Table 7: Data for Example 03

		1	2	3	4	5
	1/λi	1000	100	1000	100	1000
O4-A	1/µi	100	10	100	10	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O4-B	1/µi	100	1	100	1	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	100	100	100	100	

Table 8: Data for Example 04

		1	2	3	4	5
	1/λi	1000	100	1000	100	1000
O5-A	1/µi	100	10	100	10	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O5-B	1/µi	100	1	100	1	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	20	20	20	20	

Table 9: Data for Example 05

		1	2	3	4	5
	1/λi	1000	10	1000	10	1000
O6-A	1/µi	100	1	100	1	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O6-B	1/µi	100	0,1	100	0,1	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	100	100	100	100	

Table 10: Data for Example 06

		1	2	3	4	5
	1/λi	1000	10	1000	10	1000
O7-A	1/µi	100	1	100	1	100
	ei	0,9091	0,9091	0,9091	0,9091	0,9091
O7-B	1/µi	100	0,1	100	0,1	100
	ei	0,9091	0,9901	0,9091	0,9901	0,9091
	Ui	1	1	1	1	1
	C	20	20	20	20	

Table 10: Data for Example 07

		Q1	Q2	Q3	Q4	X
O2-A	simulation	70,1026	55,9205	44,0543	29,8837	0,7794
	+/-	0.0288	0.0321	0.0322	0.0295	0.001
	FM-method	71,2159	56,3639	43,6361	28,7841	0,7836
	error(%)	**1,59**	**0,79**	**-0,95**	**-3,68**	**0,54**
	TM-method	71,2159	56,3639	43,6361	28,7841	0,7836
	error(%)	**1,59**	**0,79**	**-0,95**	**-3,68**	**0,54**
	BM-method	71,2159	56,3639	43,6361	28,7841	0,7836
	error(%)	**1,59**	**0,79**	**-0,95**	**-3,68**	**0,54**
O2-B	simulation	51,3855	67,5935	32,3939	48,6054	0,8463
	+/-	0.0401	0.0384	0.0356	0.0395	0.001
	FM-method	57,1079	60,8907	39,1093	42,8921	0,8691
	error(%)	**11,14**	**-9,92**	**20,73**	**-11,75**	**2,69**
	TM-method	54,45	65,9087	34,0913	45,55	0,8531
	error(%)	**5,96**	**-2,49**	**5,24**	**-6,29**	**0,80**
	BM-method	55,0305	65,3195	34,6805	44,9695	0,8551
	error(%)	**7,09**	**-3,36**	**7,06**	**-7,48**	**1,04**

Table 11: Comparison of results obtained using the three decomposition methods (example 02)

		Q1	Q2	Q3	Q4	X
O3-A	simulation	15,2804	11,6633	8,3391	4,7193	0,7035
	+/-	0.0040	0.0046	0.0047	0.0040	0.0001
	FM-method	15,4962	11,7634	8,2366	4,5038	0,7043
	error(%)	**1,41**	**0,86**	**-1,23**	**-4,57**	**0,11**
	TM-method	15,4962	11,7634	8,2366	4,5038	0,7043
	error(%)	**1,41**	**0,86**	**-1,23**	**-4,57**	**0,11**
	BM-method	15,4962	11,7634	8,2366	4,5038	0,7043
	error(%)	**1,41**	**0,86**	**-1,23**	**-4,57**	**0,11**
O3-B	simulation	13,2128	12,1022	7,9016	6,7890	0,7925
	+/-	0.0040	0.0046	0.0047	0.0040	0.0001
	FM-method	14,4319	12,0051	7,9949	5,5681	0,8002
	error(%)	**9,23**	**-0,80**	**1,18**	**-17,98**	**0,97**
	TM-method	13,1693	12,9563	7,0437	6,8307	0,7903
	error(%)	**-0,33**	**7,06**	**-10,86**	**0,61**	**-0,28**
	BM-method	13,8088	12,4648	7,5352	6,1917	0,7948
	error(%)	**4,51**	**3,00**	**-4,64**	**-8,80**	**0,29**

Table 12: Comparison of results obtained using the three decomposition methods (example 03)

		Q1	Q2	Q3	Q4	X
O4-A	simulation	74,7480	46,1760	53,8280	25,2571	0,8126
	+/-	0.0238	0.0319	0.0309	0.0232	0.0001
	FM-method	74,9408	44,2766	55,7234	25,0592	0,8422
	error(%)	**0,26**	**-4,11**	**3,52**	**-0,78**	**3,64**
	TM-method	75,0821	47,4969	52,5031	24,9179	0,8112
	error(%)	**0,45**	**2,86**	**-2,46**	**-1,34**	**-0,17**
	BM-method	75,9892	45,7324	54,2676	24,0108	0,8212
	error(%)	**1,66**	**-0,96**	**0,82**	**-4,93**	**1,06**
O4-B	simulation	51,1637	67,8633	32,0907	48,7842	0,8466
	+/-	0.0404	0.0374	0.0387	0.0412	0.0001
	FM-method	55,6205	58,2596	41,7404	44,3795	0,8937
	error(%)	**8,71**	**-14,15**	**30,07**	**-9,03**	**5,56**
	TM-method	54,5165	65,773	34,227	45,4835	0,8534
	error(%)	**6,55**	**-3,08**	**6,66**	**-6,77**	**0,80**
	BM-method	55,2581	65,1271	34,8729	44,7419	0,8558
	error(%)	**8,00**	**-4,03**	**8,67**	**-8,29**	**1,09**

Table 13: Comparison of results obtained using the three decomposition methods (example 04)

		Q1	Q2	Q3	Q4	X
O5-A	simulation	16,8825	9,0199	10,9793	3,1188	0,7335
	+/-	0.0026	0.0035	0.0034	0.0025	0.0001
	FM-method	17,2350	9,6660	10,3340	2,7650	0,7561
	error(%)	**2,09**	**7,16**	**-5,88**	**-11,34**	**3,08**
	TM-method	16,5024	10,5658	9,4342	3,4976	0,7185
	error(%)	**-2,25**	**17,14**	**-14,07**	**12,15**	**-2,04**
	BM-method	17,0828	9,7992	10,2008	2,9172	0,7408
	error(%)	**1,19**	**8,64**	**-7,09**	**-6,46**	**1,00**
O5-B	simulation	13,4480	11,7896	8,2100	6,5531	0,7943
	+/-	0.0047	0.0049	0.0050	0.0047	0.0001
	FM-method	16,1597	10,1316	9,8684	3,8403	0,8417
	error(%)	**20,16**	**-14,06**	**20,20**	**-41,40**	**5,97**
	TM-method	13,1953	12,9359	7,0641	6,8047	0,7905
	error(%)	**-1,88**	**9,72**	**-13,96**	**3,84**	**-0,48**
	BM-method	14,1474	12,2048	7,7952	5,8526	0,7974
	error(%)	**5,20**	**3,52**	**-5,05**	**-10,69**	**0,39**

Table 14: Comparison of results obtained using the three decomposition methods (example 05)

		Q1	Q2	Q3	Q4	X
O6-A	simulation	75,6018	44,4480	55,5827	24,4007	0,8183
	+/-	0.0206	0.0315	0.0315	0.0217	0.0001
	FM-method	77,6128	22,4455	12,7167	22,3405	0,8441
	error(%)	**2,66**	**-49,50**	**-77,12**	**-8,44**	**3,15**
	TM-method	75,6783	46,0243	53,9757	24,3217	0,8149
	error(%)	**0,10**	**3,55**	**-2,89**	**-0,32**	**-0,42**
	BM-method	76,9229	43,3896	56,6104	23,0771	0,8279
	error(%)	**1,75**	**-2,38**	**1,85**	**-5,42**	**1,17**
O6-B	simulation	51,1807	68,0961	31,9514	48,8575	0,8466
	+/-	0.0412	0.0382	0.0372	0.0398	0.0001
	FM-method	59,1549	40,8451	4,5587	40,8042	0,8924
	error(%)	**15,58**	**-40,02**	**-85,73**	**-16,48**	**5,41**
	TM-method	54,5232	65,7593	34,2407	45,4768	0,8535
	error(%)	**6,53**	**-3,43**	**7,16**	**-6,92**	**0,82**
	BM-method	55,2822	65,1084	34,8916	44,7178	0,8558
	error(%)	**8,01**	**-4,39**	**9,20**	**-8,47**	**1,09**

Table 15: Comparison of results obtained using the three decomposition methods (example 06)

		Q1	Q2	Q3	Q4	X
O7-A	simulation	17,3206	8,0099	11,9863	2,6766	0,7478
	+/-	0.0022	0.0037	0.0037	0.0020	0.0001
	FM-method	17,1621	7,5786	12,4214	2,8379	0,8279
	error(%)	**-0,92**	**-5,38**	**3,63**	**6,03**	**10,71**
	TM-method	16,6217	10,3491	9,6509	3,3783	0,7207
	error(%)	**-4,04**	**29,20**	**-19,48**	**26,22**	**-3,62**
	BM-method	17,4054	8,8296	11,1704	2,5946	0,7624
	error(%)	**0,49**	**10,23**	**-6,81**	**-3,06**	**1,95**
O7-B	simulation	13,4455	11,9941	8,0078	6,5554	0,7946
	+/-	0.0047	0.0051	0.0051	0.0047	0.0001
	FM-method	16,5876	3,8073	3,0933	3,3807	0,8546
	error(%)	**23,37**	**-68,26**	**-61,37**	**-48,43**	**7,55**
	TM-method	13,1979	12,9339	7,06613	6,8021	0,7905
	error(%)	**-1,84**	**7,84**	**-11,76**	**3,76**	**-0,52**
	BM-method	14,1740	12,1892	7,8108	5,8260	0,7975
	error(%)	**5,42**	**1,63**	**-2,46**	**-11,13**	**0,36**

Table 16: Comparison of results obtained using the three decomposition methods (example 07)

		1	2	3	4	5	6	7	8	9	10
	1/λi	1000	100	1000	100	1000	100	1000	100	1000	100
C2-A	1/μi	100	10	100	10	100	10	100	10	100	10
	ei	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91
C2-B	1/μi	100	1	100	1	100	1	100	1	100	1
	ei	0,91	0,99	0,91	0,99	0,91	0,99	0,91	0,99	0,91	0,99
a) 200 pallets	Ui	1	1	1	1	1	1	1	1	1	1
b) 500 pallets	Ci	100	100	100	100	100	100	100	100	100	100

Table 17: Data for Example C2

		1	2	3	4	5	6	7	8	9	10
	1/λi	1000	100	1000	100	1000	100	1000	100	1000	100
C3-A	1/μi	100	10	100	10	100	10	100	10	100	10
	ei	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91	0,91
C3-B	1/μi	100	1	100	1	100	1	100	1	100	1
	ei	0,91	0,99	0,91	0,99	0,91	0,99	0,91	0,99	0,91	0,99
a) 40 pallets	Ui	1	1	1	1	1	1	1	1	1	1
b) 100 pallets	Ci	20	20	20	20	20	20	20	20	20	20

Table 18: Data for Example C3

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C2-A a)	simulation	22,7983	17,1911	22,7843	17,1499	22,8216	17,2218	22,8178	17,1845	22,8087	17,2143	0,7324
	+/-	0,0194	0,0174	0,0196	0,0172	0,0195	0,0171	0,0194	0,0171	0,0195	0,0171	0,0001
	FM-method	21,6486	18,3514	21,6486	18,3514	21,6486	18,3514	21,6486	18,3514	21,6486	18,3514	0,7873
	error(%)	**-5,04**	**6,75**	**-4,98**	**7,01**	**-5,14**	**6,56**	**-5,12**	**6,79**	**-5,09**	**6,61**	**7,50**
	TM-method	22,7769	17,2231	22,7769	17,2231	22,7769	17,2231	22,7769	17,2231	22,7769	17,2231	0,7023
	error(%)	**-0,09**	**0,19**	**-0,03**	**0,43**	**-0,20**	**0,01**	**-0,18**	**0,22**	**-0,14**	**0,05**	**-4,11**
	BM-method	22,7548	17,2452	22,7548	17,2452	22,7548	17,2452	22,7548	17,2452	22,7548	17,2452	0,7315
	error(%)	**-0,19**	**0,31**	**-0,13**	**0,56**	**-0,29**	**0,14**	**-0,28**	**0,35**	**-0,24**	**0,18**	**-0,12**
C2-B a)	simulation	12,2519	27,7603	12,2525	27,7481	12,2544	27,7496	12,2560	27,7551	12,2421	27,7299	0,7752
	+/-	0,0157	0,0253	0,0160	0,0263	0,0153	0,0252	0,0160	0,0264	0,0159	0,0263	0,0001
	FM-method	10,8735	29,1265	10,8735	29,1265	10,8735	29,1265	10,8735	29,1265	10,8735	29,1265	0,8635
	error(%)	**-11,25**	**4,92**	**-11,25**	**4,97**	**-11,27**	**4,96**	**-11,28**	**4,94**	**-11,18**	**5,04**	**11,39**
	TM-method	14,3216	25,6784	14,3216	25,6784	14,3216	25,6784	14,3216	25,6784	14,3216	25,6784	0,7367
	error(%)	**16,89**	**-7,50**	**16,89**	**-7,46**	**16,87**	**-7,46**	**16,85**	**-7,48**	**16,99**	**-7,40**	**-4,97**
	BM-method	14,3742	25,6258	14,3742	25,6258	14,3742	25,6258	14,3742	25,6258	14,3742	25,6258	0,7457
	error(%)	**17,32**	**-7,69**	**17,32**	**-7,65**	**17,30**	**-7,65**	**17,28**	**-7,67**	**17,42**	**-7,59**	**-3,81**

Table 19: Comparison of results obtained using the three decomposition methods (example C2 with 200 pallets)

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C2-A b)	simulation	54,7314	45,2234	54,7689	45,2424	54,7865	45,2879	54,7245	45,2901	54,7214	45,1971	0,7764
	+/-	0,0381	0,0352	0,0378	0,0354	0,0382	0,0353	0,0383	0,0352	0,0382	0,0354	0,0001
	FM-method	55,5121	44,4879	55,5121	44,4879	55,5121	44,4879	55,5121	44,4879	55,5121	44,4879	0,8471
	error(%)	**1,43**	**-1,63**	**1,36**	**-1,67**	**1,32**	**-1,77**	**1,44**	**-1,77**	**1,44**	**-1,57**	**9,11**
	TM-method	55,3700	44,6300	55,3700	44,6300	55,3700	44,6300	55,3700	44,6300	55,3700	44,6300	0,7780
	error(%)	**1,17**	**-1,31**	**1,10**	**-1,35**	**1,07**	**-1,45**	**1,18**	**-1,46**	**1,19**	**-1,25**	**0,21**
	BM-method	56,5711	43,4289	56,5711	43,4289	56,5711	43,4289	56,5711	43,4289	56,5711	43,4289	0,8025
	error(%)	**3,36**	**-3,97**	**3,29**	**-4,01**	**3,26**	**-4,10**	**3,37**	**-4,11**	**3,38**	**-3,91**	**3,36**
C2-B b)	simulation	38,2415	61,7988	38,2195	61,7727	38,2253	61,7610	38,2156	61,7744	38,2177	61,7733	0,8093
	+/-	0,0359	0,0359	0,0338	0,0340	0,0350	0,0353	0,0353	0,0353	0,0341	0,0342	0,0001
	FM-method	39,9312	60,0688	39,9312	60,0688	39,9312	60,0688	39,9312	60,0688	39,9312	60,0688	0,8966
	error(%)	**4,42**	**-2,80**	**4,48**	**-2,76**	**4,46**	**-2,74**	**4,49**	**-2,76**	**4,48**	**-2,76**	**10,79**
	TM-method	42,3780	57,6220	42,3780	57,6220	42,3780	57,6220	42,3780	57,6220	42,3780	57,6220	0,8089
	error(%)	**10,82**	**-6,76**	**10,88**	**-6,72**	**10,86**	**-6,70**	**10,89**	**-6,72**	**10,89**	**-6,72**	**-0,05**
	BM-method	42,6655	57,3345	42,6655	57,3345	42,6655	57,3345	42,6655	57,3345	42,6655	57,3345	0,8158
	error(%)	**11,57**	**-7,22**	**11,63**	**-7,18**	**11,62**	**-7,17**	**11,64**	**-7,19**	**11,64**	**-7,19**	**0,80**

Table 20: Comparison of results obtained using the three decomposition methods (example C2 with 500 pallets)

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C3-A a)	simulation	5,2126	2,7898	5,2143	2,7836	5,2135	2,7918	5,2121	2,7842	5,2201	2,7913	0,6182
	+/-	0,0081	0,0052	0,0078	0,0052	0,0082	0,0051	0,0083	0,0052	0,0078	0,0058	0,0001
	FM-method	4,5679	3,4321	4,5679	3,4321	4,5679	3,4321	4,5679	3,4321	4,5679	3,4321	0,6241
	error(%)	**-12,37**	**23,02**	**-12,40**	**23,30**	**-12,38**	**22,94**	**-12,36**	**23,27**	**-12,49**	**22,96**	**0,95**
	TM-method	4,2686	3,7314	4,2686	3,7314	4,2686	3,7314	4,2686	3,7314	4,2686	3,7314	0,4560
	error(%)	**-18,11**	**33,75**	**-18,14**	**34,05**	**-18,12**	**33,66**	**-18,10**	**34,02**	**-18,23**	**33,68**	**-26,24**
	BM-method	4,5177	3,4823	4,5177	3,4823	4,5177	3,4823	4,5177	3,4823	4,5177	3,4823	0,5552
	error(%)	**-13,33**	**24,82**	**-13,36**	**25,10**	**-13,35**	**24,73**	**-13,32**	**25,07**	**-13,46**	**24,76**	**-10,19**
C3-B a)	simulation	3,9285	4,0639	3,9325	4,0689	3,9355	4,0720	3,9304	4,0661	3,9326	4,0695	0,6986
	+/-	0,0029	0,0036	0,0028	0,0035	0,0027	0,0032	0,0027	0,0032	0,0027	0,0033	0,0001
	FM-method	4,0558	3,9492	4,0558	3,9492	4,0558	3,9492	4,0558	3,9492	4,0558	3,9492	0,8451
	error(%)	**3,24**	**-2,82**	**3,14**	**-2,94**	**3,06**	**-3,02**	**3,19**	**-2,87**	**3,13**	**-2,96**	**20,97**
	TM-method	3,6239	4,3761	3,6239	4,3761	3,6239	4,3761	3,6239	4,3761	3,6239	4,3761	0,4802
	error(%)	**-7,75**	**7,68**	**-7,85**	**7,55**	**-7,92**	**7,47**	**-7,80**	**7,62**	**-7,85**	**7,53**	**-31,26**
	BM-method	3,6706	4,3294	3,6706	4,3294	3,6706	4,3294	3,6706	4,3294	3,6706	4,3294	0,5223
	error(%)	**-6,56**	**6,53**	**-6,66**	**6,40**	**-6,73**	**6,32**	**-6,61**	**6,48**	**-6,66**	**6,39**	**-25,24**

Table 21: Comparison of results obtained using the three decomposition methods (example C3 with 40 pallets)

		Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	X
C3-A b)	simulation	11,1826	8,8198	11,1787	8,8123	11,1978	8,8196	11,1913	8,8214	11,1924	8,8112	0,6514
	+/-	0,0145	0,0124	0,0145	0,0123	0,0147	0,0125	0,0145	0,0124	0,0144	0,0123	0,0001
	FM-method	10,8702	9,1298	10,8702	9,1298	10,8702	9,1298	10,8702	9,1298	10,8702	9,1298	0,7132
	error(%)	**-2,79**	**3,51**	**-2,76**	**3,60**	**-2,93**	**3,52**	**-2,87**	**3,50**	**-2,88**	**3,62**	**9,49**
	TM-method	10,4084	9,5916	10,4084	9,5916	10,4084	9,5916	10,4084	9,5916	10,4084	9,5916	0,5411
	error(%)	**-6,92**	**8,75**	**-6,89**	**8,84**	**-7,05**	**8,75**	**-7,00**	**8,73**	**-7,00**	**8,86**	**-16,93**
	BM-method	10,8527	9,1473	10,8527	9,1473	10,8527	9,1473	10,8527	9,1473	10,8527	9,1473	0,6543
	error(%)	**-2,95**	**3,71**	**-2,92**	**3,80**	**-3,08**	**3,72**	**-3,03**	**3,69**	**-3,04**	**3,81**	**0,45**
C3-B b)	simulation	8,8632	11,1369	8,8626	11,1362	8,8629	11,1355	8,8632	11,1372	8,8640	11,1383	0,7159
	+/-	0,0045	0,0045	0,0045	0,0044	0,0045	0,0045	0,0046	0,0045	0,0044	0,0044	0,0045
	FM-method	10,0460	9,9541	10,0460	9,9541	10,0460	9,9541	10,0460	9,9541	10,0460	9,9541	0,8451
	error(%)	**13,35**	**-10,62**	**13,35**	**-10,61**	**13,35**	**-10,61**	**13,35**	**-10,62**	**13,33**	**-10,63**	**18,05**
	TM-method	9,4953	10,5047	9,4953	10,5047	9,4953	10,5047	9,4953	10,5047	9,4953	10,5047	0,5660
	error(%)	**7,13**	**-5,68**	**7,14**	**-5,67**	**7,14**	**-5,66**	**7,13**	**-5,68**	**7,12**	**-5,69**	**-20,94**
	BM-method	9,5967	10,4033	9,5967	10,4033	9,5967	10,4033	9,5967	10,4033	9,5967	10,4033	0,6190
	error(%)	**8,28**	**-6,59**	**8,28**	**-6,58**	**8,28**	**-6,58**	**8,28**	**-6,59**	**8,27**	**-6,60**	**-13,54**

Table 22: Comparison of results obtained using the three decomposition methods (example C3-B with 100 pallets)